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Abstract:

In this paper the much celebrated Harrod-Domar model is extended to include a non-consumable capital good. Here, growth rate of capital is directly proportional to saving rate and inversely proportional to weighted harmonic mean of capital output ratios of two sectors. Moreover, our formula includes differential prices for the two goods. Further, here, flexible prices or variable capital output ratio for consumer goods sector help to balance savings and investments avoiding the famed knife-edge problem. Our model can provide explanations for possible relationships between wealth income ratios on one side, and interest rate and rent on the other, and help to confirm the possibilities of Piketty's well-known empirical observations.

JEL Codes: E10, E22, O41

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1 Introduction

As it is well-known, the Harrod–Domar model (Harrod (1939), Domar (1946)) is the long-run dynamic version of Keynes' short run macroeconomic model. The model is widely used in development planning to determine the requirements of resources to achieve a given rate of economic growth. It is also useful to design policies to raise resources and study the effects of technology.

In the Harrod-Domar model, the growth rate of output is the ratio of national saving rate to the fixed capital output ratio. This requires the equality of the national savings rate, which depends on consumers' behavior, and the product of the growth rate of labor(a demographic-sociological fact) and the technologically fixed capital output ratio. Thus, in this model, a stable equilibrium at the maximal output level is like a knife edge. Swan (1956) and Solow (1956) extended this model to address the knife-edge issue.

Models with one good, one representative agent and a central planner are useful to study technologically feasible consumption and savings decisions, and hence, technical efficiency and inter-generational equity. However, Solow (1953) noted,

> '...in the passage from a many-commodity to a one-commodity world, considerable analytic richness is lost, and many interesting varieties of price and output behavior disappear. To mention just one example, no consideration of the role of relative prices is possible......A certain amount of depth could be added by assuming the existence of two goods differentiated with respect to their utility in consumption and in investment.'

When an input and an output is aggregated into one good, and a consumer and producer is combined as a representative agent, coordination of production, consumption, savings and investment decisions do not arise. If decisions are decentralized, production takes time, and inputs and outputs are distributed through market mechanisms, models with a minimum of two goods- an input and an output- and two agents- a producer and a consumer- and three time periods-past, present and future (for book, replacement and market values of capital) - are needed.

In this paper, we generalize the Harrod-Domar model into two sectors, one 'consumption goods' sector and another 'non-consumable capital goods' sector. We show that the growth rate is directly proportional to the saving ratio and inversely proportional to the weighted harmonic mean of the capital output ratios of two sectors. The generalized growth formula involves relative prices. However, here, for a given real interest rate, cost minimization determines product prices, the real wage rate and the investment demand, independent of consumer preferences. Again, the knife edge problem can reappear, but, with flexible product prices,the problem can have solutions other than choosing production techniques. Flexible prices, in addition to variable capital output ratios, help to balance savings and investment. Equilibrium exists for any savings rate. Equilibrium capital stock is an *increasing* function of the interest rate. Growth rate is *inversely* related to real wage and *positively* related to interest rate, supporting Piketty's empirical evidence. When savings is a non-decreasing function of the interest rate, an equilibrium real interest rate exists. If savings varies also with income, market prices may differ from producer prices resulting in profit or unemployment, idle capacity and loss in the short run. Since capital is both an input and an output for capital goods sector, the demand for capital in the capital goods sector does not depend on the price of capital. Time lags in production and durability of capital lead to compounding costs and discounting revenues, involve interest rate in the measurements and determining marginal products of capital, and make user cost of capital a non-linear function of prices involving flow and stock prices. Importantly, in contrast to a host of criticisms, Piketty's positive relation between high returns on capital and high wealth income ratios is one of the possibilities in this model as the model include relative prices and time lags. Our model can provide explanations for the relationships between wealth income ratios and interest rates for various economies in different stages of development.

2 Two Sector Growth Model

There are two goods, a consumption good (Q) and a capital good (K). Production of consumption and capital goods requires capital and labor. Technology consists of a set of techniques (α, β) for the consumption good. Capital labor substitution is characterized by the function $\beta = h(\alpha)$ where h is strictly convex and decreasing. Capital is produced with only one technique with fixed input coefficients. For each technique, the input requirements per unit of output are given by the matrix:

Table 1: Technology

	Q	K
L	α	γ
K	β	λ

Capital associated with all production techniques depreciates at an exponential rate δ. We assume constant returns to scale.

There are many techniques of production for the consumption good allowing capital labor substitution. Choice of a cost minimizing technique depends on the relative price labor and capital. In our model with fixed capital, the complete production and distribution process requires three periods. Both capital and the consumption good require one period for production. Consumption good requires capital good as an input. Hence it takes two periods to increase the output of consumption good which is available only at the end of the second period, i.e., the beginning of third period.Thus, short, medium and long runs could be defined as one, two and three or more periods, respectively. The three periods are suitable to study: (1) exchange of a given quantities of commodities or retail markets; (2) short run production and distribution with fixed capital or Keynesian macro models and (3) long run production, distribution and growth with capital accumulation.

Let us define

w = wage rate

ρ = interest rate

n = growth rate of labor

g_t = $\frac{K_{t+1}-K_t}{K_t}$ = growth rate of capital

p_c = price of consumption good.

p_k = price of capital good.

q = consumer good output per worker

y = income per worker

k = capital per worker

2.1 Relative Prices

One unit of capital requires γ units of labor and λ units of capital. Hence[1],

$$p_k = (1+\rho)\,\gamma\, w + \lambda(\rho+\delta)p_k$$

$$\Rightarrow \quad p_k = \frac{(1+\rho)\,\gamma\, w}{\{1-\lambda(\rho+\delta)\}} \tag{1}$$

and

$$p_c = (1+\rho)\,\alpha\, w + \beta(\rho+\delta)\, p_k.$$

Note the following: (1) p_k is an increasing function of w and ρ; (2) the total labor (in current labor unit adjusted for different time periods using interest rate) required to produce one unit of capital is

[1] See Appendix for equivalent derivations of this result as book and marktet values of capital and selling options.

$\dfrac{p_k}{(1+\rho)\,w} = \dfrac{\gamma}{1-\lambda(\rho+\delta)}$; (3) the share of capital, $s_{k,k}$, in capital goods sector is, $\lambda(\rho+\delta)$ and (4) when $\rho = 0, p_k = \dfrac{\gamma w}{1-\lambda\delta}$ and when $\rho+\delta \to \frac{1}{\lambda},\ p_k \to\ \infty$.

The relative price of consumption and capital good is

$$\frac{p_c}{p_k} = \frac{(1+\rho)\,\alpha\,w}{p_k} + \beta(\rho+\delta) = \frac{\alpha}{\gamma} + (\rho+\delta)\alpha\left\{\frac{\beta}{\alpha} - \frac{\lambda}{\gamma}\right\}. \tag{2}$$

when $\rho+\delta \to \dfrac{1}{\lambda},\ \dfrac{p_c}{p_k} \to\ \dfrac{\beta}{\lambda}$. Thus the range for $\dfrac{p_c}{p_k}$ is $(\dfrac{\beta}{\lambda}, \infty)$.

Using the definition of p_c, we have another expression for the relative price $\dfrac{p_k}{p_c}$,

$$\frac{p_k}{p_c} = s_{k,c} \times \frac{1}{\beta} \times \frac{1}{(\rho+\delta)} \tag{3}$$

where the share of capital in the consumption goods sector is $s_{k,c} = \dfrac{\beta(\rho+\delta)p_k}{p_c}$. All three ratios in equation(3) are determined using readily available data.

Substituting for p_k using equation(1) , p_c can be expressed as

$$p_c = \frac{(1+\rho)\,\alpha\left\{1+(\rho+\delta)\,\gamma\left(\dfrac{\beta}{\alpha} - \dfrac{\lambda}{\gamma}\right)\right\}w}{1-\lambda(\rho+\delta)}. \tag{4}$$

When $\rho+\delta \to \frac{1}{\lambda},\ p_c \to\ \infty$. Hence the range for p_c is $(0, \infty)$.

Note that the relative prices, $\dfrac{p_c}{p_k}$ and $\dfrac{p_c}{w}$, are determined as functions of the rate of interest; the technical coefficients, α and β, are also functions of the interest rate which are chosen to minimize costs (see Technical Efficiency and Consumer Goods sections). Thus, for a given ρ, a ray is fixed in the positive cone of the three dimensional price space of (p_c, p_k, w). As shown in the following sections, the interest rate is determined when the savings and production functions are specified.

Prices and income can be defined in terms of a numeraire commodity such as labor or the consumption good. For example, if labor is the numeraire and w =1, then the unit for prices and income is labor hours.

2.2 Savings, Investment and the Harrod-Domar Formula for Two Sectors

The Harrod-Domar growth formula for our model with different capital output ratios for the consumption goods and capital goods sector is generalized as follows. Let the shares of capital used for production of consumption and capital goods be denoted by η_c and $\eta_k = 1 - \eta_c$ respectively. Then, the gross output at time t is given by $Y_t = \left\{ \frac{\eta_c \, p_c}{\beta} + \frac{\eta_k \, p_k}{\lambda} \right\} K_{t-1}$. Output of the capital good sector is $\frac{\eta_k \, K_{(t-1)}}{\lambda}$. In equilibrium, gross investment is equal to output of the capital goods sector. Hence, $(g + \delta) K_{t-1} = \frac{\eta_k \, K_{(t-1)}}{\lambda}$. Hence $\eta_k = \lambda (g + \delta)$.

The fraction of income saved is $s(\rho, y)$ where y is the income per worker. The gross investment is $(g_{t-1} + \delta) K_{t-1} = K_t - K_{t-1} + \delta K_{t-1}$. Then, gross savings = gross investment implies,

$$s(\rho, y) \; Y_t = s(\rho, y) \left\{ \frac{\eta_c \, p_c}{\beta} + \frac{\eta_k \, p_k}{\lambda} \right\} K_{t-1} = (K_t - K_{t-1} + \delta K_{t-1}) \, p_k \tag{5}$$

or

$$\frac{s(\rho, y)}{g_{t-1} + \delta} = \frac{p_k}{(\eta_c p_c + \eta_k p_k)} \frac{(\eta_c p_c + \eta_k p_k)}{\left\{ \frac{\eta_c \, p_c}{\beta} + \frac{\eta_k \, p_k}{\lambda} \right\}}$$

or

$$\boxed{\frac{s(\rho, y)}{g_{t-1} + \delta} = \frac{p_k}{(\eta_c p_c + \eta_k p_k)} \, \beta_H} \tag{6}$$

where $\beta_H = \frac{(\eta_c p_c + \eta_k p_k)}{\left\{ \frac{\eta_c \, p_c}{\beta} + \frac{\eta_k \, p_k}{\lambda} \right\}}$ is the *weighted harmonic mean* of the two sectoral capital output ratios with weights, $\eta_c p_c$ and $\eta_k p_k$.

Since growth rate is a target variable, it is useful to rewrite the formula for the growth rate as

$$\boxed{g_{t-1} + \delta = \frac{s(\rho, y) \; \left\{ \eta_c \frac{p_c}{p_k} + \eta_k \right\}}{\beta_H}} \tag{7}$$

The gross investment $(g_{t-1} + \delta) \; K_{t-1} \, p_k$ = $(K_t - K_{t-1} + \delta K_{t-1}) p_k$. The savings-investment relation in (5) can also be written as

$$s(\rho, y) \left\{ \frac{\eta_c \, p_c}{\beta} + \frac{\eta_k \, p_k}{\lambda} \right\} K_{t-1} = (g_{t-1} + \delta) \; K_{t-1} \, p_k.$$

Substituting for η_c and η_k using the relation $\eta_k = \lambda(g+\delta)$ and rearranging, we get,

$$\boxed{g_{t-1} + \delta = \frac{s(\rho, y)}{[1 - s(\rho, y)]\,\beta\,\dfrac{p_k}{p_c} + s(\rho, y)\lambda}} \tag{8}$$

Note that $\eta_k = \lambda(g+\delta)$ is < 1. Another interesting relation between growth and functional income distribution is obtained by substituting for p_k in the equation(8) using equation(1).

$$\boxed{g_{t-1} + \delta = \frac{s(\rho, y)}{(1 - s(\rho, y))\dfrac{(1+\rho)\beta\gamma\, w}{\{1 - \lambda(\rho+\delta)\}p_c} + s(\rho, y)\lambda}} \tag{9}$$

2.3 Wealth Income Ratio

The wealth income ratio is $\dfrac{p_k\, k_t}{y_t}$. Since savings is equal to output of the capital goods sector,

$$s(y_t, \rho) y_t = \frac{p_k\, \eta_k\, k_{t-1}}{\lambda} = \frac{p_k\, \lambda(g_{t-1}+\delta)k_{t-1}}{\lambda} = p_k\,(g_{t-1}+\delta)k_{t-1} = \frac{p_k\,(g_{t-1}+\delta)k_t}{1+g_{t-1}}$$

$$\boxed{\Rightarrow \frac{p_k\, k_t}{y_t} = \frac{s(y_t,\rho)(1+g_{t-1})}{g_{t-1}+\delta} = \frac{s(y_t,\rho)(1+\rho)}{\rho+\delta} = s(y_t,\rho)\left\{1 + \frac{1-\delta}{\rho+\delta}\right\}} \tag{10}$$

2.4 Technical Efficiency

For given ρ, α can be chosen to minimize the unit cost p_c using (4). Choose α to minimize $\alpha[1 - \lambda(\rho+\delta)] + (\rho+\delta)\gamma\, h(\alpha)$. The necessary condition is

$$h'(\alpha) = \frac{d\beta}{d\alpha} = -\ \frac{[1 - \lambda(\rho+\delta)]}{\gamma\,(\rho+\delta)}. \tag{11}$$

Hence $h''\,\dfrac{\partial\alpha}{\partial\rho} = \dfrac{1}{\gamma\,(\rho+\delta)^2}$ or $\dfrac{\partial\,\alpha}{\partial\,\rho} = \dfrac{1}{h''\,\gamma\,(\rho+\delta)^2} > 0$. Hence, $\dfrac{\partial\,\beta}{\partial\,\rho} = h'\,\dfrac{\partial\alpha}{\partial\rho} < 0.$

When the rate of interest ρ increases, α increases and β decreases due to capital labor substitution.

The efficiency condition (11) is the relation between the marginal rate of capital labor substitution and the wage rental ratio. Our formulation using the relation between capital output ratio β and labor output ratio α, i.e., $\beta = h(\alpha)$, is related to the usual neoclassical production function relating output labor ratio q and capital labor ratio k as follows. For fixed Q, i.e. along an isoquant, $d\beta = \frac{dK}{Q}$ and $d\alpha = \frac{dL}{Q}$ and hence,

$\frac{d\beta}{d\alpha} = \frac{dK}{dL}$. Hence the efficiency condition is $\dfrac{dK}{dL} = -\dfrac{[1-\lambda(\rho+\delta)]}{\gamma\,(\rho+\delta)}$.

It is interesting to note that the relation between marginal rate of substitution and the wage rental ratio in the long run, when capital and interest rates are variable, is different from the usual ratio, $\frac{w}{(\rho+\delta)}$, in which p_k is not included since capital is fixed in the short run and hence p_k may be taken as constant. The long run wage rental ratio is $\dfrac{w}{(\rho+\delta)p_k} = \dfrac{w}{\dfrac{(\rho+\delta)\,\gamma w}{[1-\lambda(\rho+\delta)]}}$. The fact that p_k also depends on the interest rate in the wage rental ratio is not taken into account may result in biased estimates of the elasticity of substitution.

From(4), we see that p_c is an increasing function of ρ except for the term $\frac{\beta}{\alpha}$ which decreases with ρ due to capital labor substitution. If the capital labor substitution is large, this effect can dominate and lead to a negative relation between the interest rate and the consumer goods price.

The relation between the real wage $\dfrac{w}{p_c}$ and the rate of interest ρ in condition (4) is the factor price frontier. The relation may be direct or inverse depending on the capital labor ratios in the two sectors.

2.5 Employment

The full employment condition is given by

$$L_t = \left\{\eta_{ct}\,\frac{\alpha_t}{\beta_t} + \lambda(g_t+\delta)\frac{\gamma}{\lambda}\right\} K_t \tag{12}$$

The capital labor ratio at time t is

$$k_t = \frac{K_t}{L_t} = \frac{1}{\left\{\eta_c\,\dfrac{\alpha}{\beta} + \eta_k\dfrac{\gamma}{\lambda}\right\}} = \frac{1}{\left\{\eta_c\,\dfrac{\alpha}{\beta} + (g_t+\delta)\,\gamma\right\}}$$

Note that the aggregate capital labor ratio $k_t = \dfrac{1+g_{t-1}}{1+n}\,k_{t-1}$ is a share weighted harmonic mean of the sectoral capital labor ratios, $\dfrac{\beta}{\alpha}$ and $\dfrac{\lambda}{\gamma}$. If capital grows at a constant rate g, then, $k_t = \left\{\dfrac{1+g}{1+n}\right\}^t k_0$.

The long run equilibrium capital stock per worker k_t, is given by

$$k_t = \frac{1}{\dfrac{\alpha}{\beta} - \lambda(g_t+\delta)\left\{\dfrac{\alpha}{\beta} - \dfrac{\gamma}{\lambda}\right\}} \tag{13}$$

We will interpret the relations as long run equilibrium conditions and include time subscripts only when it is needed for clarity. This includes the possibility of steadily growth with different rates for capital and labor.

2.6 Consumer Goods

Consumption good supply per worker is given by

$$q^s = \frac{Q_t}{L_t} = \frac{1}{L_t}\,\frac{\eta_c K_{t-1}}{\beta} = \frac{1}{1+n}\,\frac{1}{\left\{\eta_c\,\frac{\alpha}{\beta} + (g_{t-1}+\delta)\,\gamma\right\}}\,\frac{\eta_c}{\beta}$$

Choose α to maximize q^s. The necessary and sufficient condition is

$$\frac{d\beta}{d\alpha} = -\,\frac{\eta_c}{\gamma\,(g_{t-1}+\delta)} \tag{14}$$

Also, q^s can be written as

$$q^s = \frac{\{1-\lambda(g+\delta)\}}{(1+n)\alpha\left\{1-\lambda(g+\delta)\left\{1-\dfrac{\beta\gamma}{\lambda\alpha}\right\}\right\}}$$

Note that $\dfrac{dq^s}{dg} = \dfrac{-\,\gamma}{(1+n)\beta\left\{\dfrac{\alpha}{\beta} - \lambda(g+\delta)\left\{\dfrac{\alpha}{\beta}-\dfrac{\gamma}{\lambda}\right\}\right\}^2} < 0$; hence q^s is a decreasing function of g.

Also,

$$\frac{d^{\,2}q^s}{dg^2} = \frac{-\,2\lambda\gamma\left\{\dfrac{\alpha}{\beta}-\dfrac{\gamma}{\lambda}\right\}}{\left\{\dfrac{\alpha}{\beta} - \lambda(g+\delta)\left\{\dfrac{\alpha}{\beta}-\dfrac{\gamma}{\lambda}\right\}\right\}^3}$$

If $\frac{\lambda}{\gamma} > \frac{\beta}{\alpha}$, then q^s is a strictly concave function of $g+\delta$. In the opposite case, q^s is a strictly convex function of g. See Figure 1. The demand for consumption goods is $q^d = \dfrac{(1-s(y,\rho)Y}{p_c}$ and the market clearing condition for consumer goods market is

$$Q^d = \frac{(1-s(y,\rho)Y_t}{p_c} = Q^s = \frac{L_t\{1-\lambda(g+\delta)\}}{(1+n)\alpha\left\{1-\lambda(g+\delta)\left\{1-\dfrac{\beta\gamma}{\lambda\alpha}\right\}\right\}} \tag{15}$$

From the efficiency conditions (11) and (2.6), we see that per worker consumption is a maximum when $\hat{\rho} = g_{t-1}$. The gross investment rate is

$$\boxed{\hat{\rho} + \delta = g_{t-1} + \delta} \tag{16}$$

From (4), the cost minimizing (efficient) real wage rate is

$$\boxed{\frac{\hat{w}}{\hat{p_c}} = \frac{1 - \lambda(\hat{\rho} + \delta)}{(1 + \hat{\rho})\, \hat{\alpha} \left\{ 1 + (\hat{\rho} + \delta)\, \gamma \left(\dfrac{\hat{\beta}}{\hat{\alpha}} - \dfrac{\lambda}{\gamma} \right) \right\}} = \frac{\hat{q}^s (1 - s_{kk})}{(1 + \hat{\rho}) \left(1 + s_{kk} \left\{ \frac{\hat{\beta}\gamma}{\hat{\alpha}\lambda} - 1 \right\} \right)}} \tag{17}$$

where $\hat{q}^s$ is the consumption goods output per worker and $\hat{\alpha}$ and $\hat{\beta}$ are functions of $\hat{\rho}$.

2.7 Long Run Steady State Equilibrium

The long run is defined as the period in which the supply of capital can be increased. In our model, it is three periods. A long run equilibrium is one in which the three markets, consumer goods, capital goods and the labor market clear. For a given rate of interest, relative prices are determined and consumer goods supply is based on full employment of labor. Hence, when the consumer goods market clears, the labor market will also clear as can be seen from equation(12). It is a steady state long run equilibrium if the growth rate is the same as the population growth rate. In this paper we study only steady state equilibria.

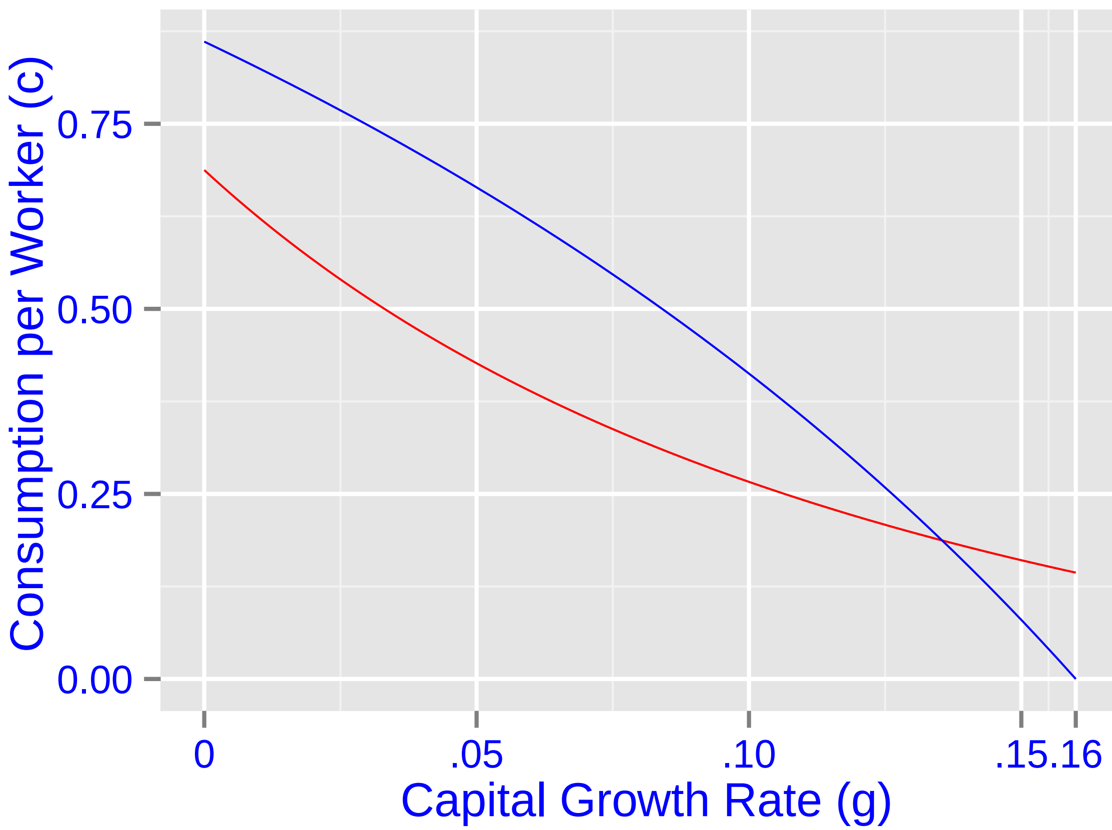

Figure 1: Consumption per Worker and Growth Trade Off

Data:

Depreciation (delta) = 0.04

Blue Curve: $\beta_1 = 3, \alpha_1 = 1, \lambda_1 = 5, \gamma_1 = 1$

Red Curve: $\beta_2 = 5, \alpha_2 = 1, \lambda_2 = 3, \gamma_1 = 2$

National income identity, which holds for all prices, is the Walras‘s law. It is :

$$y_t = \left[\frac{p_c}{\beta}(1 - \lambda[g_{(t-1)} + \delta]) + p_k(g_{(t-1)} + \delta)\right] k_{(t-1)} \tag{18}$$

When any two markets clear, the third market also clears. Since the consumer goods and labor markets are linked, it is enough if we show that the capital market clears. In other words, if there is an interest rate equating savings and investment, then by Walras‘s law both consumer goods and the labor market will also clear.

The long run equilibrium conditions are given by equations (8, 12,2.6,2.7). In a steady state, $g_t = g_{t-1} =$

g^*.

From (8) and (2.6) we have the following equation in ρ.

$$\boxed{\rho^* + \delta = \frac{s(\rho^*, y^*)}{(1 - s(\rho^*, y^*))\, \beta^* \, \dfrac{p_k}{p_c} + s(\rho^*, y^*)\lambda} < \frac{1}{\lambda}} \tag{19}$$

where y^* is the national income defined in Table 2 below.
The rate of interest ρ^* is based on the following three conditions: 1. Cost minimization or efficiency condition (11); 2. maximal consumption for given growth rate (2.6); and 3. savings constraint or the generalized Harrod-Domar formula (8), while $\hat{\rho}$ is based only on the first two.

The relation in (2.7) is an implicit non-linear function of ρ^*. A sufficient condition for existence of a solution is that the savings function is bounded away from $\frac{1}{\lambda}$ say by $\bar{s} = 1 - \epsilon$ for some $\epsilon > 0$. The right hand side of equation (2.7) is a continuous function over the compact interval $[0, \frac{1}{\lambda}]$. By Brower's fixed point theorem, there is a fixed point, ρ^*. Since the function is bounded above by $\dfrac{\bar{s}}{\lambda}$, $\rho^* + \delta < \frac{1}{\lambda}$ and hence, both p_k^* and p_c^* will be $< \infty$.

Long run equilibrium relations of macro economic variables are given in the Table 2. below. It is interesting to note that g^* is a strictly increasing function of the the rate of interest ρ^* and that gross investment per worker, i^*, is also a strictly *increasing function of ρ^* irrespective of difference between the capital labor ratios in the two sectors unlike shinkai(1960) and Uzawa(1964)*. (To verify, substitute for k^* and divide both numerator and denominator by $\lambda(g^* + \delta)$)

For a savings rate of 15% and capital output ratios of 3 and 4, and labor out ratios of 1.5 and 1 respectively for the consumer and capital goods sectors and a depreciation rate of 4%, the growth rate is 3.5%. The corresponding rate of interest is 5.7%.

The wealth income ratio reduces to the capital output ratio as in the Harrod-Domar model with one good (see equation(2.7)). This follows if $\lambda = \beta$ and $\alpha = \gamma$ and no time lag in production. Then, $p_k = p_c$ and $\frac{k^*}{y^*} = \beta^*$. Since there is no time lag, $1 + g^*$ is not included in the formula. Since the capital output ratio is inversely related to the rate of interest, wealth income ratio as used by Piketty should be *negatively* related to the rate of interest and not positively related as observed by Piketty. The inclusion of two goods in our model brings in prices and helps to explain the observed relation of aggregate wealth and income. The wealth income ratio in our model is an increasing function of the savings ratio and a decreasing function of the rate of interest(see equation(10)). Savings depend on income and the interest rate. The effect of interest rate on income depends on the prices and the outputs and may be negative or positive. Thus the relation between wealth income ratio and the interest rate could be positive or negative.

Table 2: Long Run Steady State Equilibrium Relations

Variable	Formula
Interest Rate ($\rho^* = g^*$)	$\dfrac{1}{\left\{\dfrac{[1-s(\rho^*,y^*)]}{s(\rho^*,y^*)}\dfrac{\beta^*}{[1-\lambda(\rho^*+\delta)]\frac{\alpha^*}{\gamma}+[\lambda(\rho^*+\delta)\frac{\beta^*}{\lambda}]}+\lambda\right\}}-\delta$
Capital/Labor (k^*)	$\dfrac{1}{\dfrac{\alpha^*}{\beta^*}+\lambda(g^*+\delta)\left\{\dfrac{\gamma}{\lambda}-\dfrac{\alpha^*}{\beta^*}\right\}}$
Gross Investment/Labor (i^*)	$p_k^*(g^*+\delta)k^*$
Consumption/Labor (q^*)	$p_c^*\dfrac{\{1-\lambda(g^*+\delta)\}\,k^*}{\beta^*}$
Income/Labor (y^*)	$\left[\dfrac{p_c^*}{\beta^*}(1-\lambda(g^*+\delta))+p_k^*(g^*+\delta)\right]k^*$
Wealth/Income Ratio (p$_k^*k^*/y^*$)	s(y*, ρ^*) $\left\{1+\dfrac{1-\delta}{\rho^*+\delta}\right\}$

The last two rows in Table 2 are the following two equations in two variables, y^* and ρ^* if we substitute ρ^* by g^*.

$$\frac{k^*p_k^*}{y^*}=\frac{1+\rho}{\dfrac{p_c^*}{p_k^*}\dfrac{1}{\beta^*}\{1-\lambda(\rho+\delta)\}+\lambda(\rho+\delta)\dfrac{1}{\lambda}} \tag{20}$$

and

$$\frac{k^*p_k^*}{y^*}=s(y^*,\rho^*)\left\{1+\frac{1-\delta}{\rho^*+\delta}\right\} \tag{21}$$

The two conditions are independent since the first is based on cost minimization and the second relates to equilibrium in the capital goods market involving savings function.

The right hand side of Condition(2.7) is the price weighted *harmonic mean* of the equilibrium capital output ratios. Due to production lags, current income is from the previous period's wealth (capital stock).

Multiplication by $1+\rho$ transforms the aggregate capital output ratio to the ratio of wealth and income of the same period. The effect of rate of interest on the aggregate wealth income ratio depends on the net effect of the following three effects: i. Change in β^* depending on the elasticity of substitution of capital by labor in the consumption goods sector; ii. The growth effect in the numerator due to the lag structure and the growth effect in the denominator arising from the change in the shares of capital for both sectors (note $g^* = \rho^*$); and iii. change in the relative price $\frac{p_c}{p_k}$. Since we assume that the capital goods sector is more capital intensive than the consumption goods sector and since the share of capital for the capital goods sector, $\lambda(\rho+\delta)$ increases, the first two effects will increase the aggregate capital output ratio with an increase in ρ. The direction of change in the relative price $\frac{p_c}{p_k}$ is ambiguous. The slope of the KY curve is more likely to be positive. The effect of any shift in this curve will depend on whether it intersects the IS curve from below or above. In Figure.2, we have assumed that the growth effect dominates the price effect and the substitution effects, in case they move in the opposite direction.

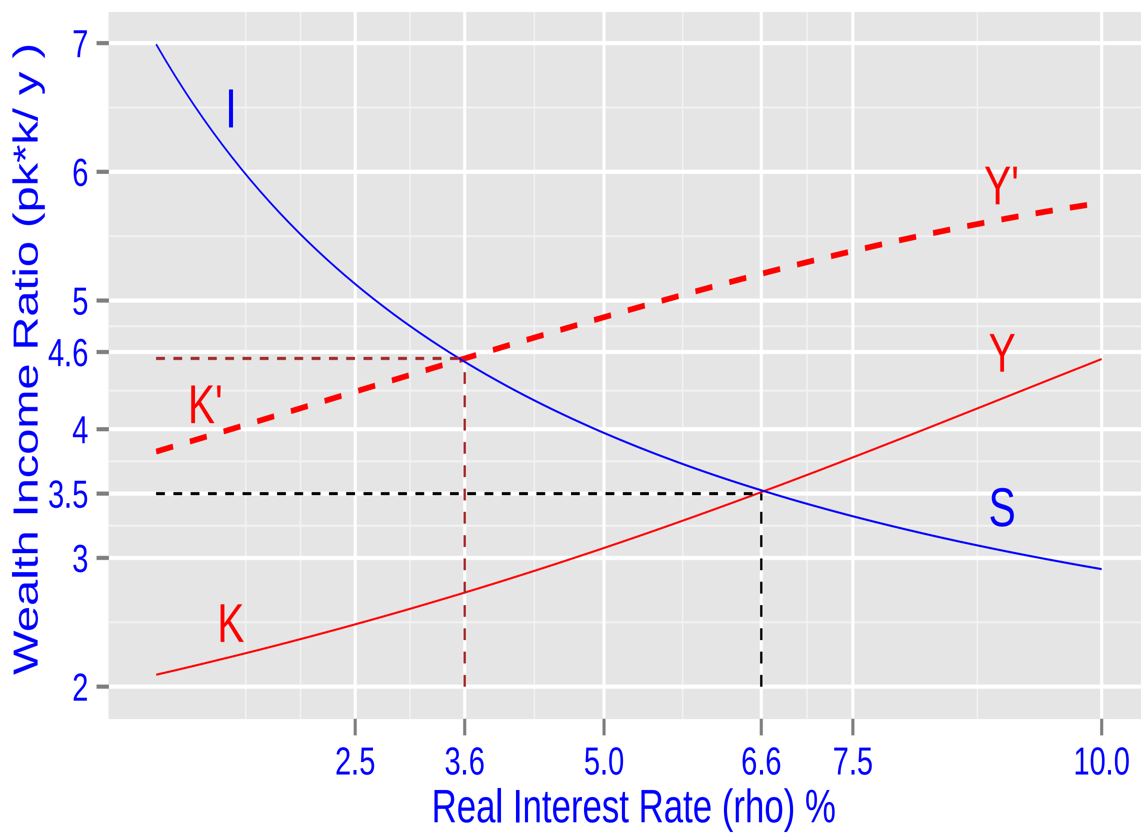

Figure 2: Wealth Income Ratio and the Rate of Interest

Data:

Depreciation (delta) = 0.04

Blue Curve (IS): savings rate 15%

Red Curve (KY): $\beta_1 = 3, \alpha_1 = 2, \lambda_1 = 5, \gamma_1 = 1$

3 Discussion

3.1 Results

The one good Harrod-Domar model with fixed capital labor and savings ratios is of immense importance, but the assumptions are too restrictive. Variable capital labor ratios in the Solow-Swan model eliminated the knife-edge problem of the model. Inclusion of a non-consumable capital good and market mechanisms for decentralized decisions enhances the richness of the model by involving relative prices and provides room for public policies. Flexible prices in our model helps to prove the existence of equilibrium for any savings ratio. A capital good which is different from the consumption good and is used both as an input and output (in the capital goods sector) makes the long run investment function an *increasing* function of the rate of interest in contrast to the Keynesian short run demand for investment based on marginal efficiency of capital decreasing with the rate of interest and also brings out the limiting role of capital for growth. Time lags in production leading to costs and revenues distributed over time result in compounded book values and discounted market values of capital, and make user cost of capital a non-linear function of interest rate, depreciation and the endogenous value of capital. Since capital is both an input and an output for the producers, the effect of fiscal and monetary policies on the economy become more complicated. For example, increasing tax on capital may increase the supply of capital.

Mahalanobis (1953) two sector growth model is a limited extension of Harrod-Domar model. Though it has a capital good and a consumption good, the techniques of production are fixed and, being a planning model, market prices are absent. More importantly, the demand side is not included. In our model, capital good sector plays a relatively important role by limiting the growth rate. The capital output ratio in the capital good sector, λ provides an upper bound, $\frac{1}{\lambda}$, for the growth rate . Maximizing per capita income in the Mahalanobis model is not same as maximizing per capita consumption in our model as is evident from the per capita consumption formula in Table 2. The room for trade off between growth and consumption, in our model, can help to design monetary and fiscal policies to promote growth with equity. This is a substantive difference of our paper from that of Mahalanobis.

Shinkai (1960)and Uzawa (1964) two sector models are based on the assumption that all wages are consumed and all profits saved (see Solow (1953)) for the restrictive nature of this assumption) while our savings function varies with income and the rate of interest. Secondly, Shinkai and Uzawa models are continuous time models with instantaneous production and without lags and, therefore, financing needs. On the other hand we have discrete time and production lags.

In our paper, zero profit implies competition. Our model fits well with the current thinking that consumption, and not income, is the appropriate variable for long run optimization. Growth rate and per capita consumption are inversely related. Ours is a long run equilibrium and not necessarily a steady state since population growth rate need not equal the growth rate of capital.

In the short run, keeping interest rate low will help to increase employment and output when there is excess capacity. It is Keynesian. Interest rates have been low in the U.S. since 2008, and Japan even before. But is it good in the long run? Now it is seven years since USA is sticking to the policy of almost zero interest rate regime. Is it not long run? In our model, in long run high interest rate is desirable as it is positively linked with growth (though not with consumption). It will increase saving, making funds available for investment and hence growth and reduce relative capital use in consumption good sector thereby releasing capital for capital good sector and helping in higher growth. It can also increase the production of capital good as its output price increases though price of input also increases.

Piketty emphasizes wealth income ratio , i.e. , capital output ratio. He remarks it has increased significantly, but neoclassical economists disagree since higher level of capital must imply lower return on capital. Piketty's observation is that both capital and return on capital are high. This is theoretically consistent with our model as opposed to his critics due to the following three reasons.: 1. time lags in production and durability of capital with compounding costs and discounting revenues; 2. extension of the model to include two goods and their prices and 3. capital as an input in its own production. Further we observe that the capital stock to consumption goods ratio is even higher than the wealth income ratio. It is not inconsistent with neoclassical theory of decreasing marginal productivity of capital in the consumption goods sector, since producers choose the output mix to maximize income.

Let us look at Figure (2). The IS curve, which is the equilibrium condition for the capital goods market, is downward sloping, i.e., wealth income ratio is a decreasing function of interest rate. On the other hand the KY curve, which relates aggregate capital output ratio to the interest rate, is upward sloping, i.e., the price weighted aggregate capital output ratio is an increasing function of the interest rate. If KY curve shifts upward, then a high wealth income ratio with low interest rate is obtained, and that corresponds to the present day US economy. The KY curve can shift upwards due to various reasons including a rise in $\lambda or \beta$. Piketty observes high wealth income ratio and high return to capital. As is well known, return to capital is the combination of interest, profit and rent. It is entirely possible that low interest rate occurs along with a very high rent, making Piketty's observation consistent with our model.

3.2 Limitations

In our model, though the production and consumption decisions are separated, the mechanisms of transfer of savings to investors or transfer of capital income to consumers are not explicitly specified. There is no financial asset for lending and borrowing resources. The model needs to be extended to include bonds, money and taxes for financing private and public investment. Other important limitations are exclusion of human capital and factors such as land and technical progress.

3.3 Extensions

Our model must be extended to include a bond market and the Government. It will help to study public investment, effects of fiscal and monetary policies such as efficient mix of taxes and debt equating forgone consumption at the margin. Multiple equilibria are possible and Government policy may be required to choose welfare improving equilibria when they exist. Our forthcoming paper which introduces land, bond market and Government will make these issues more explicit. This paper will also attempt to generalize the static IS-LM curve into dynamic framework which will include a KY curve. It will also be interesting to study the effects of rent and profits on investment and consumption behavior. Disaggregating capital goods sector into physical and human capital will help to study their contrasting behavior in the model, for example, while physical capital depreciates, human capital may appreciate due to gain of experience (or learning by doing).

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Appendix

Measures of Capital

1. **Book Value**

Cost of one unit (say a \$) of capital is $(1+\rho)\gamma\, w + \lambda\,(\rho+\delta)$. Cost of $\lambda\,(\rho+\delta)$ of capital, in the preceding period, is $\lambda\,(\rho+\delta)\{(1+\rho)\gamma\, w + \lambda\,(\rho+\delta)\}$= $\{\lambda\,(\rho+\delta)\}\,\{(1+\rho)\gamma\, w\} + \{\lambda^2\,(\rho+\delta)^2\}$. Repeating this process, we have the infiite series of costs: $\{(1+\rho)\gamma\, w\}\,\{1+\lambda\,(\rho+\delta)+\lambda^2\,(\rho+\delta)^2+...\}+\lambda^n\,(\rho+\delta)^n$. Since the last term goes to 0, the limit of this sum is $\dfrac{(1+\rho)\gamma\, w}{1-\lambda(\rho+\delta)}$ implying $p_k = \dfrac{(1+\rho)\gamma\, w}{1-\lambda(\rho+\delta)}$.

2. **Market Value**

The present market value of one unit of capiatal in the capital goods sector is calculated as follows. Start with one unit of capital. Since capital depreciates at the rate δ, the sequence of discounted net revenues available from the one unit of capital is ,
$\dfrac{1}{\lambda}\left\{\dfrac{p_k}{1+\rho}-\gamma\, w\right\}\left\{1+\dfrac{1-\delta}{1+\rho}+\left\{\dfrac{1-\delta}{1+\rho}\right\}^2+...\left\{\dfrac{1-\delta}{1+\rho}\right\}^n ..\right\} = \dfrac{p_k-(1+\rho)\gamma\;w}{\lambda\,(\rho+\delta)}$. Under competition, zero

profit implies, $p_k = \dfrac{p_k - (1+\rho)\gamma\ w}{\lambda\,(\rho+\delta)} \Rightarrow p_k = \dfrac{(1+\rho)\gamma\, w}{1-\lambda(\rho+\delta)}$.

3. **One Period Futures Market for Capital**

In the context of an appropriate price theory for the Leontief dynamical input-output system, Solow(1953) provided the following simple interpretation. With one unit of capital, consider the following two options: i. Sell now for a price of p_k. ii. Use the capital with $\dfrac{\gamma}{\lambda}$ units of labor to produce $\dfrac{1}{\lambda}$ units of capital with a net output of $\dfrac{1}{\lambda}\left\{\dfrac{p_k - \gamma\, w\,(1+\rho)}{(1+\rho)}\right\}$ and sell the remaining capital for $\dfrac{(1-\delta)\,p_k}{1+\rho}$. Both options should have the same present value. Hence, $p_k = \dfrac{p_k - \gamma\, w\,(1+\rho)}{\lambda\,(1+\rho)} + \dfrac{(1-\delta)\,p_k}{1+\rho} \Rightarrow p_k = \dfrac{(1+\rho)\gamma\, w}{1-\lambda(\rho+\delta)}$.

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