Identifying the Vulnerable

Concepts and Measurement

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ABSTRACT

Vulnerability to poverty, which broadly captures the susceptibility to becoming poor in the future, has become an integral part of any deprivation assessment. In this paper we take a fresh look at measuring vulnerability, where we separate out the identification part of whether an individual is vulnerable, from the aggregation part which measures the level of vulnerability of the individual. In doing so, we also take into account the substantial informational challenges that one faces in the context of measuring vulnerability. These challenges lead us to propose identification rules which are not too demanding in terms of information. We axiomatically characterise several identification rules and a widely used measure of vulnerability. This allow us to have a deeper insight into normative judgements behind those criteria. Further, we provide an empirical illustration of the identification and measurement rules proposed in this paper using real world data from Bangladesh.

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1   Introduction

Vulnerability to poverty, which captures susceptibility of households and individuals to poverty in the future, is fast becoming an integral part of any deprivation assessment. The Human Development Report (UNDP, 2014, p.10) noted that "Vulnerability threatens human development and unless it is systematically addressed, by changing policies and social norms, progress will be neither equitable nor sustainable." It was followed by the UN Sustainable Development Goals in 2015, which suggested as an integral part of poverty reduction, the implementation of a "..nationally appropriate social protection systems and measures for all, ... , and by 2030 achieve substantial coverage of the poor and the vulnerable."[1] While these goals to reduce vulnerability recognise the uncertainty individuals face in their daily livelihoods, successful policy design and implementation requires effective identification of who is vulnerable and what their level of vulnerability is. Since vulnerability is associated with uncertainties regarding future outcomes and their sources, both the identification and the measurement of vulnerability is likely to be a complex and informationally demanding exercise.

This paper puts forth a systematic way of addressing these two distinct questions of (a) identifying the vulnerable, and (b) measuring the level of vulnerability that each individual faces. Based on axiomatic justifications, we propose some reasonable identification rules to address the vital question of who should be considered as vulnerable. In recent years although there has been significant development in our understanding of how to measure and estimate vulnerability (see Calvo (2018); Gallardo (2017) and Fuji (2016) for comprehensive reviews), several empirical papers have used adhoc rules for identifying the vulnerable (Hohberg et al. 2018). In our proposed identification rules we explicitly take into consideration the informational constraints that are inherent in the nature of vulnerability. Once we identify who the vulnerable are, we are faced with a second non-trivial problem of measuring the level of vulnerability to poverty for each of those individuals, since we have to 'aggregate' all the possible future levels of income of the individual in a reasonable manner. In our second contribution to the literature, we axiomatically characterise a class of measures which is widely used in many empirical papers to measure the vulnerability of individuals. Hence our paper can be viewed as providing a theoretical underpinning to much of the applied work in the context of vulnerability.

Informational issues are crucial when it comes to measuring vulnerability which by definition is an ex-ante concept that requires one to 'estimate' the poverty status of an individual in the future, given the information today.[2] Recent studies such as, Dang and Lanjouw (2017), Chakravarty (2016), and de la Fuente et al. (2015) have used a vulnerability line, which is similar in concept to a poverty line,

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[1]More details about the Sustainable Development Goals can be found at https://sustainabledevelopment.un.org/sdgs

[2]There is also a strong literature where vulnerability is defined as low expected utility in the future (Ligon and Schecter, 2003)

to identify the vulnerable. However, the vulnerability line is operationalised using ex-post data and therefore ignores the ex-ante nature of vulnerability. More importantly, the uncertainty with respect to future poverty also means that the vulnerable need not be just a broader set of the poor since some of the poor of today may be able to escape poverty tomorrow and join the ranks of non-poor. Hence, defining the vulnerable as those who are certain to be poor in the future (Angelilo, 2014) may not be very useful since it would miss out on a substantial number of individuals who may have a high probability of falling into poverty in the future but not with full certainty. On the other hand, identifying an individual as vulnerable to poverty if in future there is any possibility of the individual falling below the poverty (or reference) line (Calvo and Dercon, 2013, Dutta, Foster, Mishra 2011, and Glewwe and Hall, 1998) suffers from the undesirable possibility of considering individuals as vulnerable who may be really rich in general but in certain rare situations, may actually fall into poverty.

A slightly different route to identifying the vulnerable is taken by many empirical studies such as Ward (2016), Imai et al. (2011), Jha and Dang (2010), Zhang and Wan (2008), Christiaensen and Subbarao (2005), Chaudhuri (2003), Chaudhuri et al. (2002), where the level of vulnerability for each individual is first computed and then those above a minimum threshold are identified as vulnerable. Apart from this arbitrary setting of the vulnerability threshold, this approach also suffers from the intuitively unsatisfactory position of calculating individual vulnerability levels before identifying whether the individual is vulnerable or not.[3] As a result, identification of the vulnerable becomes informationally as demanding as measuring their level of vulnerability since we need to collect detailed income information on every individual to estimate their vulnerability, which can be expensive. For instance, in estimating poverty in developing countries, which is typically based on expenditure, regular information on household consumption becomes so cumbersome to collect that for most purposes, governments implement a scorecard test to identify and target poor households. In the case of India, for example, the government collects consumption information on a regular basis for around two hundred thousand households in a country of 1.2 billion people, which is less than 0.001 percent of the population. Scaling up is extremely expensive both in money and time (Sharan, 2018). When it comes to vulnerability, which is about the future, the informational challenges around it can be even more substantial.[4]

Following Sen (1976, 1979), we view the identification of who is vulnerable as a distinct and different exercise than measuring the level of vulnerability of individuals. From that perspective, there is no reason why the informational burden has to be same for identifying the vulnerable as it is for measuring their level of vulnerability. A policy maker, for instance, may decide to have a

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[3] This point has been made clear in the context of poverty measurement in Sen (1981, Chapter 3).

[4] The data issues with respect to vulnerability assessment have been discussed in detail in Patel et al. (2017).

broader and to some extent a looser set of criteria to identify the vulnerable and yet may undertake a different exercise using detailed information to measure the level of vulnerability of those identified as vulnerable. Similar to the direct approach of identifying the poor (Sen 1979, 1981), we could consider information on just the type and frequency of shocks impacting individuals to identify the vulnerable.[5] This is typically the information that is collected for vulnerability assessments, (see Erman et al. (2018), Gerlitz et al. (2014), Korboe (2011), World Bank (2007)), where the focus is on the exposure to different hazards that people experience. Still, we may need a more detailed set of information which includes the associated loss in income from the different adverse shocks before we can reasonably measure the level of vulnerability for each individual.

Our proposed approach follows the steps of some of the recent literature in the area of poverty and deprivation which address the issue of identification explicitly. Alkire and Foster (2011), in the context of multi-dimensional deprivation, identify an individual as multi-dimensionally deprived based on the number of dimensions the individual is deprived in. This is known as the counting approach.[6] In this paper, we demonstrate that a similar framework can be quite appropriately used to identify the vulnerable based on the number of adverse shocks a person is exposed to. These adverse income shocks can be related to future states of nature an individual may face such as sudden illness or death (health shocks), and extreme weather events (natural shocks). Studies such as Knippenberg and Hoddinott (2017), McCarthy et al. (2016), Vargas Hill and Porter, (2016) and Heltburg and Lund (2009) have used such information on discrete shocks to examine vulnerability. Thus, if an individual had become poor when faced with unexpected income shocks due to ill health or due to natural hazards such as flooding, we can capture these impacts as outcomes of future states of the world that the individual may face. These future states of the worlds, which capture adverse income shocks, are independent of each other. The greater the exposure to these adverse shocks that lead to poverty, the higher will be the vulnerability of the individual to poverty.

Using this natural notion of vulnerability to poverty which arises out of adverse income shocks, we propose a cut-off based on a weighted sum of the number states the individual is likely to be deprived in the future as a way to identify the vulnerable. There can be several variants of this approach depending on the nature of these weights. In our context, probabilities associated with these states

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[5] Sen (1979, p. 290) identifies an individual as poor under the direct method if the "..actual consumption basket happen to leave some minimum need unsatisfied." He further states (p.290), "In an obvious sense the direct method is superior to the income method, since the former is not based on particular assumptions of consumption behaviour which may or may not be accurate."

[6] In Foster (2007), an individual is identified as chronically poor based on the number of time periods they were poor. In either context, what the exact number of dimensions that the individual needs to be deprived in or what the exact number of periods the individual must be poor in to be identified as multi-dimensionally poor or chronically poor respectively, is determined by a cut-off parameter which reflects a value judgement of the policy maker or the analyst. Atkinson (2017) in his report of the monitoring of global poverty for the World Bank had recommended the counting approach as an extremely useful tool in deprivation analysis.

are taken as natural weights. However, in the absence of information on probabilities of the different states of nature, we could consider a rule with uniform probability distribution across all states. On the other hand if we have more richer set of information, we could also consider identifying rules where we take into account the level of income shortfall associated with the various income shocks. We axiomatically characterise these different identifying rules later in the paper. To the best of our knowledge, using the counting based framework has so far not been applied to the context of identifying the vulnerable.

Once we identify the vulnerable individuals, we can then apply 'aggregation' rules to compute the level of individual vulnerability. These aggregation rules can be different from the identification rules, in part driven by the different set of information one may have for identification and for aggregation. Further, in this paper, we axiomatically characterise one such widely used aggregation rule which is similar to the Foster Greer and Thorbecke (1984) (FGT) class of measures of poverty. In the context of vulnerability, this is known as the expected FGT class of vulnerability measures.[7] The advantage of this class of measures is that it fits in with our conceptual notion of the different states of nature, through which we capture the adverse shocks, as being independent of each other. So far we identified if an individual is vulnerable and if so, their level of vulnerability. We can then aggregate the information on individual vulnerability to build societal vulnerability. In this paper we consider a simple average of individual vulnerability as a measure of societal vulnerability. We demonstrate that the societal measure of vulnerability is the sum of the vulnerability arising out of the different income shocks. Thus, policymakers can easily be able to identify the key shocks leading to vulnerability to poverty in their population.

The plan of the paper is as follows. In the next section, we develop the notations and illustrate through numerical examples our proposed concept of identifying the vulnerable and measuring their level of vulnerability to poverty. In Section 3, we explore the identification rules in greater detail and discuss axioms to characterise our proposed identification rule. Section 4, provides an axiomatic characterisation of the expected FGT class of vulnerablity measures. In the following section, we discuss the societal vulnerability measure and its decomposition into different states of nature. An empirical illustration, using data from Bangladesh is provided in Section 6 of the paper. It demonstrates how we can apply the identification methods discussed in this paper in the real world. The final section draws the arguments of the paper together with some brief remarks about further research.

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[7]Apart from the studies referred to in previous paragraphs, some of the other studies that employ the FGT class of measures are Zhang and Wan (2006), Azam and Imai (2009), Imai et al. (2010). Pritchett et al. (2000) use the expected headcount in the multiperiod context.

2   Concept of Vulnerability Measure

2.1   Notation

Consider a society of N individuals.   Suppose there are m states of the world.   Each individual i, faces a lottery, which shows the probability, p_s^i and deprivation d^{is} in each state s of the world. Similar to Luce and Raiffa (1957) we represent the lottery as $L^i = (p_1^i, d^{i1}; p_2^i, d^{i2}; ...; p_m^i, d^{im})$, where $m \geq 2$ and $\forall s$, $d^{is} \in [0, 1]$, $p_s^i \geq 0$, and $\sum_{s=1}^m p_s^i = 1$.[8]   To reduce notational burden, when there is no scope of confusion, we shall drop the superscript i, from the lotteries and the associated probabilities and deprivations when dealing with an individual.

The probability vector associated with lottery L is represented as $\mathbf{p}^L = \{p_1, p_2, ..., p_m\} \in P$ where P is the set of all probability distributions over the m states.   While for each individual we consider the deprivation profile $\mathbf{d} = \{d^1, d^2, ..., d^m\}$ as given, the deprivation profile could differ between individuals.   Let $\mathcal{L}$ be the set of all such lotteries an individual may face such as $L = (p_1, d^1; p_2, d^2; ...; p_m, d^m)$, and $\widetilde{L} = (\widetilde{p}_1, d^1; \widetilde{p}_2, d^2; ...; \widetilde{p}_m, d^m)$.

In this paper, to focus on the identification part, we base on the deprivation status in each of the states with a positive probability of occurrence.   The consideration of states with positive probability implicitly assumes that states which have no likelihood of occurrence should not play a role in the determination of vulnerability of an individual.

Associated with lottery L we create a $(1 \times m)$ deprivation identification vector $\mathbf{r}^L = \{r^1, r^2, ..., r^m\}$ based on the following rule

$$\forall s, \ r^s = \begin{cases} 1 & \text{if } p_s.d^s > 0 \\ 0 & \text{otherwise} \end{cases}.$$

for all $s = 1, ..., m$.   It is clear that $\mathbf{r}^L$ basically partitions the states into deprived and the non-deprived, such that all the deprived states have a positive probability of occurrence and have been assigned the highest deprivation.

While a general identification function is defined as $\rho : \mathcal{L} \longrightarrow \mathbb{R}_+$, for our purpose we use probability weighted identification function, $\rho^E : \mathcal{L} \longrightarrow [0, 1]$, which is defined as:

Definition 1 ρ^E *is a vulnerability identification function such that for any individual (i) $\rho^E(L) > \rho^E(L')$ iff $\mathbf{p}^L \bullet \mathbf{r}^L > \mathbf{p}^{L'} \bullet \mathbf{r}^{L'}$ (ii) $\rho^E(L) = \rho^E(L')$ iff $\mathbf{p}^L \bullet \mathbf{r}^L = \mathbf{p}^{L'} \bullet \mathbf{r}^{L'}$.*

$\rho^E(L) = 1$ would mean that an individual facing lottery L, is certainly deprived in the future whereas $\rho^E(L) = 0$, that the individual is certainly not deprived.

We identify any individual facing lottery L as vulnerable if the scalar product $\mathbf{p}^L \bullet \mathbf{r}^L \geq \theta$,

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[8]In our context, suppose an individual i earns y_{is} in state s. Then deprivation in that state is given by $d^{is} = (z - y_{is})/z$, if $y_{is} \leq z$, otherwise $d^{is} = 0$, where z is the future poverty line.

where $\theta \in (0, 1]$ is the cutoff point. If $\mathbf{p}^L \bullet \mathbf{r}^L \geq 0$, every individual is deemed vulnerable and the identification rule does not have any discerning power. We refer to this as the 'universal' rule. For any meaningful identification to occur, at the least, we consider $\theta = \theta_{\min}$, where an individual is identified as vulnerable if they are poor in any future state with positive probability.[9] This is known as the 'union' approach. On the other hand when $\theta = 1$ the individual is deemed vulnerable if he is poor in all future states. Once we identify the individual as vulnerable, we then use the lottery the individual faces to come up with the level of vulnerability an individual faces. Thus for any individual i, vulnerability is measured by $V : (0, 1] \times \mathcal{L} \longrightarrow \mathbb{R}_+$. In addition if $\rho(L) < \theta$, then $V(L) = 0$.

For a society with N individuals and m future states of the world, we have a $N \times m$ vulnerability matrix which we denote as $M^{N,m}$. Each cell of the matrix shows the probability and deprivation associated with an individual in that state. Thus each row of the matrix lists the probability and deprivation of one individual over the different states which is effectively the lottery that the individual faces. Let Φ denote the set of all such matrices. The societal vulnerability is a function $V^S : \Phi \longrightarrow \mathbb{R}_+$.

2.2 Vulnerability Measure

In this section, we present our vulnerability measure in three steps and then illustrate it with an example. Suppose an individual face a lottery L. The first step focuses on identifying whether an individual is vulnerable or not and hence included in our overall vulnerability index. If the individual is identified as vulnerable from the first step i.e. $\rho(L) = \mathbf{p}^L \bullet \mathbf{r}^L \geq \theta$. When we talk about vulnerability, we often refer to the *probability that the individual will be deprived* in the future. The common intuition uses the crudest (coarsest) partition of states; deprived or not deprived. Our identification method formalizes this intuition. Even when there are several states while identifying whether an individual is vulnerable or not, we do not need to consider the extent of deprivations in different states; the total probability that an individual is likely to be in a deprived state will suffice for our purpose. We call this identification strategy the *P-rule*.

In the second step, we aggregate his deprivation across all the states and his vulnerability is given as

$$V(L) = \begin{cases} \sum_{s=1}^{m} p_s (d^s)^\alpha & \text{if } \rho(L) \geq \theta \\ 0 & \text{otherwise} \end{cases}.$$

(1)

where $\alpha \geq 0$. Note that when $\alpha = 0$, there will be no difference between the aggregation rule proposed in (1) and the *P-rule* identification strategy given by $\rho^E(L)$.

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[9] To be precise, for any lottery L, $\theta = \theta_{\min}$ means $\theta = \min(p_1, p_2, ..., p_{m'})$, where $\forall s = 1, ..., m'$, $d^s = 1$. Thus $\theta = \theta_{\min}$, refers to a specific value of θ which is the minimum probability associated with a state with positive deprivation in lottery L.

Once each individual's vulnerability has been computed, in the third step, we take a simple aggregation individual vulnerabilities to find out the societal level of vulnerability. Let $M^{N,m}$ represent the societal deprivation matrix of N individuals over m future states. Then the societal vulnerability measure can be represented as

$$V^S(M^{N,m}) = \frac{1}{N}\sum_{i=1}^{N}V(L) \qquad (2)$$

Note that the societal average does not consider the distribution of vulnerability across the population.[10] However, the simple formulation has the great advantage of being able to decompose vulnerability into different population subgroups. Hence, policymakers would be able to target vulnerable groups based on region, gender, ethnicity, or any other population-level characteristics.

2.3 Illustrated Examples

Our first two examples would focus on measuring vulnerability using different counting rules. In the first example, we use the counting rule ρ^E (see Definition 1).

Example 1 *Consider the following vulnerability matrix with $N = 3$ and $m = 3$:*

$$M^{3,3} = \begin{pmatrix} & State\ 1 & State\ 2 & State\ 3 \\ & 0.1,0.1 & 0.8,0 & 0.1,0.2 \\ & 0.2,0 & 0.1,0 & 0.7,\ 0.8 \\ & 0.5,0.6 & 0.25,0.4 & 0.25,0.2 \end{pmatrix}$$

where in each cell the first number is the probability and the second is the deprivation faced by an individual. Thus the lotteries faced by each of the individuals are, $L^1 = (0.1, 0.1; 0.8, 0; 0.1, 0.2)$, $L^2 = (0.2, 0; 0.1, 0; 0.7, 0.8)$, and $L^3 = (0.5, 0.6; 0.25, 0.4; 0.25, 0.2)$. Thus $\mathbf{r}^1 = \{1, 0, 1\}$, $\mathbf{r}^2 = \{0, 0, 1\}$, and $\mathbf{r}^3 = \{1, 1, 1\}$.

First, let us begin with Stage 1. Let the cut-off point $\theta = 0.5$. $\mathbf{p}^{L^1} = \{0.1, 0.8, 0.1\}$ and the scalar product $\mathbf{p}^{L^1} \bullet \mathbf{r}^1 = 0.2 < 0.5$. Hence individual 1 is not considered vulnerable. On the other hand $\mathbf{p}^{L^2} = \{0.2, 0.1, 0.7\}$ and $\mathbf{p}^{L^2} \bullet \mathbf{r}^2 = 0.7 > 0.5$ which implies that individual 2 is vulnerable. Similarly for the third individual, $\mathbf{p}^{L^3} = \{0.5, 0.25, 0.25\}$ and $\mathbf{p}^{L^3} \bullet \mathbf{r}^3 = 1 > 0.5$ which indicates that individual 3 is also vulnerable.

In stage 2 we calibrate individual vulnerability using. (1) and $\alpha = 1$; $V(L^1) = 0$, $V(L^2) = 0.56$ and $V(L^3) = 0.45$.

In stage 3, the overall societal vulnerability will be $V^S(M^{3,3}) = 1/3(V(L^1) + V(L^2) + V(L^3)) = 0.34$.

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[10] Basu and Nolen (2006) argues that in the context of unemployment, distribution of unemployment among the population can be useful in the overall measure of unemployment since the burden of unemployment being shared equally is desirable. Thus, unlike here, an increase in vulnerability to unemployment is preferable.

In this example, although individual 2 is deprived only in one state, compared to individual 1 who is deprived in two states, we identify individual 2 as vulnerable instead of individual 1. This is because the probability associated with the state in which individual 2 is deprived is far higher than the combined probability of the deprived states of individual 1. Thus individual 2 is more likely to be deprived in the future than individual 1.

The *P-rule* identification strategy, however, can sometimes lead to counter-intuitive results, where vulnerability rankings can change under different cut-off points represented by θ. We demonstrate this by an example below.

Example 2 *Suppose the lotteries faced by two individuals are:* $L^1 = (5/6, 0.3; 1/6, 0; 0, 0.95)$, *and* $L^2 = (0, 0.3; 2/3, 0; 1/3, 0.95)$.

For calibrating vulnerability under the counting approach, note that $\mathbf{r}^{L^1} = \mathbf{r}^{L^2} = \{1, 0, 1\}$, $\mathbf{p}^{L^1} = \{5/6, 1/6, 0\}$, *and* $\mathbf{p}^{L^2} = \{0, 2/3, 1/3\}$. *Thus,* $p^{L^1} \bullet \mathbf{r}^{L^1} = 0.83$ *and* $p^{L^2} \bullet \mathbf{r}^{L^2} = 0.33$.

For $\theta = 0.5$, $\rho(L^1) > 0.5 > \rho(L^2)$, *hence for* $\alpha = 1$, $V(L^1) = 0.25 > V(L^2) = 0$.

For $\theta = 0.3$, $\rho(L^1) > \rho(L^2) > 0.3$, *hence for* $\alpha = 1$, $V(L^1) = 0.25 < V(L^2) = 0.32$.

This reversal of ranking of vulnerability, under different values of θ, happens because of the identification property, where any individual who is not identified as vulnerable is assigned a zero value for vulnerability irrespective of their deprivation levels. This ranking reversal happens in other contexts too such as for multidimensional poverty where this issue has been highlighted by Pattanaik and Xu (2018).

One way to overcome this problem, as suggested by Pattanaik and Xu (2018), is to take into account the deprivation in each state in the identification function. This method is implicitly followed widely in the empirical literature on vulnerability where vulnerability of each individual is calibrated and then a cut-off, above which individuals are identified as vulnerable, is applied to that level of vulnerability. If the level of vulnerability of the individual is lower than the cut-off, the individual is not considered to be vulnerable. While this does not lead to 'inconsistencies', it conflates the identification and aggregation issues. If we already know what an individual's vulnerability is, the exercise of then identifying whether the person is vulnerable or not is vacuous. Further, as we have argued before, it also puts the same informational burden for identifying whether an individual is vulnerable and measuring her level of vulnerability. In many situations, such detailed information may not be available. In the following section, we discuss this issue in greater depth.

3 Identification Rules

We first discuss the general issues related to identification rules for vulnerability and then characterise an identification rule.

3.1 Inconsistency of Identification Rules

We have seen in Example 2 that the counting identification rule can lead to a reversal of vulnerability rankings of lotteries. However, this is not a unique problem with the counting rule. In fact for any identification rule which is not monotonically related to the individual vulnerability function (aggregation function), we will be able to demonstrate reversal of the vulnerability rankings.

As alluded before, if we identify any individual with $\mathbf{p}^L \bullet \mathbf{r}^L \geq 0$ as vulnerable, then the identification rule is 'universal' and all individuals are considered vulnerable. On the other hand, for other values of such as $\theta = 0.5$, the identification rule bites and some individuals may not be considered as vulnerable. Once individuals are identified as vulnerable, the level of vulnerability is then calibrated using 'aggregation' rules such as in equation (1). We identify the individual using a 'ρ^E' function (as in Definition 1) and we measure their vulnerability using the 'V' function (as in equation 1). Hence we need to ensure some consistency in identifying the vulnerable and measuring their level of vulnerability, under different values of θ, so that situations such as in Example 2 do not arise.

Suppose Q is the set of individuals who are identified as vulnerable, then we can state the following axiom:

Axiom 1 *Axiom of Consistency (A1): The identification rule ρ satisfies axiom of consistency if for any two individuals i and j with lotteries L^i and L^j respectively, $V(L^j) \geq V(L^i)$ under universal identification, then for all $\theta \geq \theta_{\min}$, $i \in Q$ implies $j \in Q$.*

The axiom of consistency essentially states that when identification is 'universal' and we find that vulnerability of individual j is greater than that of i, then it can never be the case that when identification becomes more stringent, that individual i is identified as vulnerable whereas individual j is not. More broadly if individual i has less vulnerability relatively to another individual j at lower values of θ, then it cannot be that at higher values of θ, i is identified as vulnerable whereas j is not. However, as we show below, satisfaction of this property comes at a cost, where the identification rule cannot be very different from the aggregation function. We demonstrate that unless $\rho(L)$ is a monotonic function of $V(L)$, axiom of consistency would always be violated.

Suppose $\rho(L) \neq h(V(L))$. It implies that the ordinal rankings of lotteries based on $\rho(L)$ and

$V(L)$ will be different. Consider the universal identification case, $\mathbf{p}^L \bullet \mathbf{r}^L \geq 0$. Thus, $\exists i, j$ such that

$$\rho(L^1) \leq ... \leq \rho(L^j) \leq ... \leq \rho(L^i) \leq ... \leq \rho(L^n) \tag{3}$$

and

$$V(L^1) \leq ... \leq V(L^i) \leq .. \leq V(L^j) \leq ... \leq V(L^n). \tag{4}$$

Let $\rho(L^i) > \rho(L^j)$ and $V(L^j) > V(L^i) > 0$. From (3), it is clear that $\rho(L^i) > \rho(L^j) \geq 0$. Hence, both i and j are identified as vulnerable and thus we calibrate their vulnerability, which from (4) shows $V(L^j) \geq V(L^i)$.

Now define,

$$\theta = \frac{\rho(L^i) + \rho(L^j)}{2}$$

Then from (3), $\rho(L^i) > \theta > \rho(L^j)$, which implies $i \in Q$ and $j \notin Q$, violating the axiom of consistency (A1).

Thus, the only way the consistency will be preserved is if the identification function is just a monotonic transformation of the vulnerability function (aggregation rule). One can view this as an impossibility result since it excludes the possibility of different functions for the identification and aggregation steps. However, there is no compelling reason why we should have the same function to identify individuals who are vulnerable and also measure their level of vulnerability. It may stand to reason that although related, identification and aggregation are different exercises and hence it may be justifiable to have separate functions to process these two concepts. For instance, identification of who is vulnerable may be based on a limited set of information, because all we may need is some evidence to show that they are vulnerable. Once we know who the vulnerable are, then for the aggregation part of measuring the level of vulnerability, we may need a far richer set of information which includes information on the deprivation on different states in addition to the probabilities of the states with positive deprivation.

3.2 Characterising the Identification Rule

The first axiom captures the notion that if two lotteries have only one state of positive deprivation with positive probability, then the lottery which has a higher probability of the deprived state should be ranked higher in terms of being identified as vulnerable compared to the other lottery. Thus if the lottery with the lower probability in the deprived state is identified to be vulnerable, so should the other.

Axiom 2 *Axiom of Single State Deprivation (A2): Consider a lottery L such that $p_k.r^k > 0$, and*

$\forall s \neq k,\ p_s.r^s = 0$. *Let L' be such that $p'_l.r^l > 0$, and $\forall s \neq l, p'_s.r^s = 0$. If $p'_l > p_k > 0$ $[p'_l = p_k > 0]$ then $\rho(L') > \rho(L)$ $[\rho(L') = \rho(L)]$.*

The intuition for the next axiom is quite straight forward. Consider a lottery with deprivation in k states. If we have another lottery with deprivation in an additional state besides the k states, where the probability of the additional state of deprivation has been transferred from one of the previous k deprived states. Then essentially the probability of not being deprived in the future has remained the same. Thus both lotteries should be equally identified as vulnerable or not vulnerable as the case may be. Before we state the axiom, we define the concept of probability transfer, where lottery L, with k deprived states with positive probability, is represented as L_k.

Definition 2 *We say L'_{k+1} is derived through a probability transfer from L_k where*
$L_k = (p_1, d^1; p_2, d^2; ...; p_k, d^k; p_{k+1}, d^{k+1}; ...p_m, d^m)$, *and* $L'_{k+1} = (p_1, d^1; p_2, d^2; ...; p_k - \delta, d^k; p_{k+1} + \delta, d^{k+1}; ...; p_m, d^m)$, $p_k > \delta > 0$ *and* $d^k > 0,\ d^{k+1} > 0$.

Based on this definition we can now state the axiom formally.

Axiom 3 *Axiom of Invariance to Probability Transfers (A3): Suppose L'_{k+1} is derived through a probability transfer from L_k, where $k < m$. Then $\rho(L') = \rho(L)$.*

Given these two axioms, we can show the following result.

Proposition 1 *An identification function ρ satisfies Axioms of Single State Deprivation (A2) and Axiom of Invariance to Probability Transfers (A3) iff $\rho = \rho^E$.*

Proof: See Appendix A

This, however, is not the only identification rule that can be used. While the *P-rule* identification strategy needs information on the different shocks and their frequency, depending upon the level of information that is available, we discuss two further identification rules.

Counting-rule: Suppose we just know the different shocks that individuals face, however, we have no information on the frequency of the shocks. A quick way to identify the vulnerable then might be based on counting the number of shocks. Note that in our framework the shocks are mapped on to different future states of nature. Suppose we consider four different types of shocks that individuals can suffer over the past year: natural shocks, economic shocks, health shocks, law and order shocks. Then the maximum count of shocks would be 4 and the minimum would be 1. The 'union' approach to identification would imply that anyone with atleast one count of shock would be considered vulnerable. Similarly, the intersection approach would consider anyone who has faced all the four shocks to be vulnerable. In addition, we can also have the cut-off such as two or three,

11

to identify the vulnerable. So essentially in the absence of more information, we are counting the number of shocks for identification. We provide an axiomatic framework for such a rule in Appendix B.

Max-rule: In addition to the information on the shocks and their respective frequency, suppose we have the information on the maximum deprivation that people had suffered. This maximum deprivation could take any value between $(0,1]$ and would vary between individuals since it is based on their realised deprivation in the face of all the different shocks. In such case, we can use that information to create a 'max' based identification rule, where the probabilities of shocks multiplied with the maximum deprivation is considered for the identification criteria. This is similar to the *P-rule*, where instead of each of the deprived state being given a value of one, it now is the maximum *possible* level of deprivation that each individual faces. In that case, an individual is identified as vulnerable if the probability-weighted maximum deprivation exceeds a given level. In Appendix B we provide a characterisation of such identification rule.

4   Characterising Individual's Vulnerability Measure

Our first axiom follows the equivalent of the focus axiom that has been widely put forth in Alkire and Foster (2011) and Foster (2007), in the context of multidimensional deprivation and chronic poverty. In the Alkire and Foster (2011) approach there is a cut-off based on the number of dimensions one must be poor to be considered deprived overall. In the current context, we have a cut-off based on the probability that the person is poor in the future. So for instance, consider a lottery $L = (0.15, 1; 0.2, 1; 0.1, 1; 0.25, 0; 0.2, 0)$ faced by an individual, who is deprived in the first three states only. Now if we consider the cut-off level of $\rho^E(L)$ as 0.4, then obviously the person is vulnerable. This is intuitive since the total probability that the individual will be deprived in the future is 0.45.

Axiom 4 *Focus Axiom (A4): Consider any $L = (p_1, d^1; p_2, d^2; ...; p_m, d^m)$ such that $\rho^E(L) < \theta$. Then $V(L) = 0$.*

The cutoff value θ is exogenously determined by policymakers, practitioners or analysts, based on what they think is appropriate for their context. In the empirical literature in vulnerability $\theta = 0.5$, is often considered as the cut-off value. However, there are studies such as Ward (2016) which consider a different cut-off value.

The following two axioms are similar to those of Dutta Foster and Mishra (2011). First, we define the convex combination of two lotteries as follows:

Definition 3 *Suppose $L = (p_1, d^1; p_2, d^2; ...; p_m, d^m)$ and $L^1 = (p'_1, d^1; p'_2, d^2; ...; p'_m, d^m)$. Then $\lambda L + (1-\lambda)L' = (\lambda p_1 + (1-\lambda)p'_1, d^1; \lambda p_2 + (1-\lambda)p'_2, d^2; ...; \lambda p_m + (1-\lambda)p'_m, d^m)$, where $0 < \lambda < 1$.*

The intuition for our next axiom is that any additional uncertainties that are not captured through the lotteries do not impact vulnerability. Suppose depending upon the policies government undertakes, one ends up with lottery L with probability λ or lottery L' with probability $(1 - \lambda)$. One could claim that this extra layer of uncertainty arising out of government policy can lead to more vulnerability than the vulnerabilities from the two lotteries put together. The next axiom rules out such possibilities. Hence vulnerability of a convex combination of lotteries should be the same as the convex combination of the vulnerability of each of the lotteries.

Axiom 5 *Axiom of Decomposability (A5): Consider any two deprivation lotteries L and L' such that $V(L) > 0$ and $V(L') > 0$. Then $V(\lambda L + (1 - \lambda)L') = \lambda V(L) + (1 - \lambda)V(L')$.*

The implication of this axiom would be to make the vulnerability measure linear in probabilities. It will thus generate the von Nueman-Morgenstern expected utility structure for the vulnerability measure.

The intuition for our next axiom comes from the well-known monotonicity axiom for poverty (Sen, 1976). In the context of vulnerability, it means that vulnerability should increase when the probability of occurrence of a state with higher deprivation increases relative to a state with lower deprivation. It implies that to reduce vulnerability we have to ensure that the individual becomes less likely to fall into high deprivation states.

Axiom 6 *Axiom of Monotonicity (A6): Consider two lotteries $L = (p_1, d^1; ...; p_k, d^k; ...; p_{k'}, d^{k'}; ...; p_m, d^m)$ and $L' = (p_1, d^1; ...; p_k + \delta, d^k; ...; p_{k'} - \delta, d^{k'}; ...; p_m, d^m)$, $\delta > 0$ such that $p_k > 0$; $p_{k'} > \delta$, $d^k \geq d^{k'} > 0$. Then $V(L) \leq V(L')$.*

Keeping with the previous theme of transfers of probability across states, the next axiom addresses the question on the change in vulnerability due to probability transfers when there is a reduction in deprivation across all states. This reduction in deprivation can happen due to economic development or better provision of insurance options to protect against different shocks. Consider a lottery $L = (p_1, d^1; ...; p_m, d^m)$. We denote $L_\lambda = (p_1, \lambda d^1; ...; p_m, \lambda d^m)$, $0 < \lambda \leq 1$. The formal statement of the axiom is as follows:

Axiom 7 *Axiom of Scale Invariant Transfer (A7): Consider lotteries $L = (p_1, d^1; ...; p_k, d^k; ...; p_l, d^l; ...; p_m, d^m)$, $L' = (p_1, d^1; ...; p_k + \delta, d^k; ...; p_l - \delta, d^l; ...; p_m, d^m)$, $\widetilde{L} = (\widetilde{p}_1, d^1; ...; \widetilde{p}_s, d^s; ...; \widetilde{p}_t, d^t; ...; \widetilde{p}_m, d^m)$, $\widetilde{L}' = (\widetilde{p}_1, d^1; ...; \widetilde{p}_s + \delta, d^s; ...; \widetilde{p}_t - \delta, d^t; ...; \widetilde{p}_m, d^m)$, where $d^k \geq d^l$, $d^s \geq d^t$, $p_l - \delta > 0$ and $\widetilde{p}_t - \delta > 0$. Suppose, $V(L') - V(L) = V(\widetilde{L}') - V(\widetilde{L})$, then a scale invariant transfer would imply $V(L'_\lambda) - V(L_\lambda) = V(\widetilde{L}'_\lambda) - V(\widetilde{L}_\lambda)$.*

Intuitively, if the vulnerability of two individuals is impacted to the same extent due to an adverse event, then according to the axiom of scale-invariant transfer, the extent of impact should remain the same even if deprivation for both individuals across all states, is reduced by the same proportion. For example, consider two individuals, one who faces high deprivation in case of flooding, and another for whom drought leads to very high deprivation. If an increase in these extreme weather events leads to an equivalent increase in vulnerability for both individuals, then this axiom implies that the increase in vulnerability should remain the same for both, even if the government decides to provide a better social safety net which reduces deprivation across all future states. Note that level of vulnerability might be reduced with better social safety net, but the change in vulnerability, from higher probabilities of flooding and drought, would be the same for both individuals. In other words, better social safety nets will not lead to differential impacts in terms of vulnerability faced by the individuals.

These four axioms lead us to our result that characterises the individual vulnerability measure based on the specific functional form of the deprivation function represented in (1), which is the FGT class of vulnerability measures.

Proposition 2 *A measure of vulnerability, V, of an individual satisfies Axiom of Focus (A4), Axiom of Decomposability (A5), Axiom of Monotonicity (A6), Axiom of Scale Invariant Transfer (A7) iff:*

$$V(L) = \begin{cases} \sum\limits_{s=1}^{m} p_s (d^s)^{\alpha} & \text{if } \mathbf{p}^L \cdot \mathbf{r}^L \geq \theta \\ 0 & \text{otherwise} \end{cases}, \ \alpha \geq 0.$$

Proof: See Appendix A.

Note that since $0 \leq d^s \leq 1$, and $0 \leq p_s \leq 1$, it is clear that from Proposition 2 that $0 \leq V(L) \leq 1$. Further, if $\forall s, d^s = 1$, then $V(L) = 1$, and if $\forall s, d^s = 0$, $V(L) = 0$. Interestingly, α can be interpreted as a parameter of risk aversion. When $\alpha = 1$, the individual is risk neutral. For $\alpha > 1$, the individual is risk-averse in the sense that increased spread of the deprivation distribution in the future would lead to worse outcomes, i.e. increase in vulnerability. Vulnerability is the lowest when the future deprivations are spread equally across the states. Hence higher the α, greater is the risk aversion. On the other hand for $\alpha < 1$, the individual would be considered risk-loving where a larger spread of the deprivation distribution would reduce vulnerability.

5 Societal Measure of Vulnerability

Although our main focus in this paper is on the identification part and the vulnerability at the individual level, for the purpose of completeness of the discussion, we use a simple average of individual

vulnerabilities as in (2) to capture the overall societal vulnerability. One could allow a more general functions such as a weighted average of individual vulnerabilities, or an average which takes into account the correlation among vulnerabilities (Calvo and Dercon, 2013). However, if we want to have some notion of decomposability in our societal measure, where the societal vulnerability is the sum of vulnerabilities of subpopulations, then we would be limited to a narrower set of functions (Foster and Shorrocks, 1991), which includes the simple average that we consider here.

Using the societal vulnerability measure, as given in (2), we discuss some interesting decomposition of vulnerability. From Proposition 2 we know that for any individual i, faced with lottery L^i, the vulnerability of i is given by $V(L^i) = \sum_{s=1}^{m} p_s^{L^i}(d^s)^\alpha$, if they are identified as vulnerable. Now consider the case where $\alpha = 0$. Then societal vulnerability can be written as

$$V^S = \frac{1}{N}\sum_{i=1}^{N}\sum_{s=1}^{m} p_s^{L^i}. \tag{5}$$

Let q be the number of individuals who have been identified as vulnerable then we can write (5) as

$$V^S = \frac{q}{N}.\frac{\sum_{i=1}^{q}\sum_{s=1}^{m} p_s^{L^i}}{q} = H.I_0 \tag{6}$$

where H is the head-count ratio of the vulnerable, and I_0 is the average probability with which people will fall into poverty in the future and thus reflects the intensity of the vulnerability. In this sense, the measure is very similar to the adjusted head-count ratio proposed by Alkire and Foster (2011) in the context of multi-dimensional poverty.

We can easily expand this analysis for $\alpha > 0$. In that context, we can decompose the societal vulnerability in a similar manner to (6). Thus

$$V^S = \frac{q}{N}.\frac{\sum_{i=1}^{q}\sum_{s=1}^{m} p_s^{L^i}(d^{is})^\alpha}{q} = H.I_\alpha \tag{7}$$

where I_α is the intensity of vulnerability and H is the head-count ratio. When $\alpha = 1$, I_α captures the average deprivation people will suffer in the future.

Another advantage of using the simple average of individual vulnerabilities for arriving at societal vulnerability is that we can decompose the societal vulnerability based on the different states. As we have modelled here, if the future states are based on different shocks, then we can decompose the

societal vulnerability based on the shocks. We start with societal poverty,

$$V^S = \frac{1}{N}\sum_{i=1}^{N}\sum_{s=1}^{m}p_s^{L^i}(d^{is})^\alpha,$$

which can be written as,

$$V^S = \sum_{s=1}^{m}V^{sh}, \tag{8}$$

where

$$V^{sh} = (1/N)\sum_{i=1}^{N}p_s^{L^i}(d^{is})^\alpha \tag{9}$$

is the vulnerability associated with each shock. This decomposition is particularly useful if we are interested in understanding how the different shocks contribute to the societal vulnerability.

6 Empirical Illustration

In this section, we apply the vulnerability measure developed in this paper on a data from a hamlet, Hrishipara, in Kapasia sub-district of Dhaka division, in Bangladesh for 2017 and 2018. Our data comes from the Hrishipada Daily Diary project where individuals were asked to keep a record of the daily income and expenditures. We have close to 646,000 data points, which records the almost daily financial transactions of 72 households, from May 2015 to April 2019. However not all the years had the same number of households; further not all households came under the survey at the same time and there is considerable variation in terms of the number of observations that we have for each household. In a series of articles Rutherford (2017a, 2017b, 2016a, 2016b) provided a detailed structure of the data and how one can use it to track financial transactions of the poor.[11]

For our illustration, we have chosen two years – 2017 with 71 households and 2018 with 60 households, since these years have the largest number of households surveyed for the full year. We also have the 'broad' information on household size, which is the number of people on average living in the household over the sample period. For instance, household size for household 1 is 1.1, which reflects the fact that around 10 percent of the time, there was a second person in the house.[12] The total number of individuals in our sample for 2017 and 2018 is 298 and 250 respectively. The average household size is 4.18.

Our interest here is to assess the vulnerability of individuals to income poverty. We define income as any earnings during the sample period, which includes wages from jobs and profits from businesses, loans taken and savings withdrawals, and gifts and transfer received. For the self-employed, we

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[11]More information on the financial diary project and data can be found at https://sites.google.com/site/hrishiparadailydiaries/home

[12]However for our purpose we have rounded the household size to the nearest whole number.

do subtract from their income, the legitimate expenses that one has to incur to keep their business running. For instance, we deduct the costs of the products sold for shop keepers, similarly, we deduct the cost of repairing vehicles for households who ply their vehicles to earn money. The income of the household divided by the household size gives us the individual income. To smoothen out the noise in daily income, we aggregate the daily information into weekly income. We have on average 37 weeks of income data for each of the 71 households in 2017 and 52 weeks of income data for each of the 60 households in 2018. The average weekly income for individuals in our sample is around 2400 taka (Bangladesh's currency) for 2017 and 2900 for 2018. In comparison, the GDP per capita of Bangladesh for 2017 and 2018 was USD 1516 and 1675 respectively which using current exchange rates roughly equates to a per capita weekly income of 2900 and 3200 taka.[13] Hence our sample is on average poorer than Bangladesh at large.

The income shocks that households face each year are divided into four states, with one state capturing all the positive shocks and the rest three states capturing negative shocks of varying intensity: (i) extreme shocks, (ii) moderate shocks, (iii) mild shocks. We classify the household under extreme shock if the weekly per capita income falls under \$13.3, which is equivalent to \$1.90 per day. Similarly, the moderate shock is for weekly per capita incomes between \$13.3 and \$22.4, which is equivalent to \$3.20 per day. The mild shock is when weekly per capita income is between \$3.20 per day and \$5.50 per day, which is equivalent to between \$22.4 and \$38.5. We consider any per capita income above \$38.5 per week as a state with positive shock. These are standard poverty lines considered by the World Bank (2017) in their assessment of poverty. Using current exchange rates, in terms of Bangladesh currency, individuals with weekly income not more than 1100 taka is considered to be under extreme shock, individuals with income between 1100 and 1860 taka is considered to be under moderate shock, individuals with weekly income between 1860 and 3200 taka is under mild shock and any individuals with weekly income above 3200 taka faces positive shocks.

To calibrate the deprivation in each of the different states, we take 3200 taka per week as the cut-off point. One justification is that we take 3200 taka per week to be the minimum income above which all individuals are guaranteed to be non-poor in the future. It is similar in spirit to Dang and Lanjouw (2017) where they consider the vulnerability line to be that level of income which ensures that anyone earning at least that amount will not fall into poverty in the future. We take the average income in each state and then calculate the normalised gap from the cut-off point. Therefore, for all the states with negative shocks, the deprivation will range between $[0, 1]$ and for the state with positive shocks, the deprivation would be considered as zero. What is important to note is that the states are nested in terms of the deprivation. This is because we have assumed that income below, say for

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[13]We have assumed 1 USD to be equal to 100 Bangladeshi taka.

instance, 1100 taka, will be in the extreme shock state, incomes between 1100 and 1860 taka will be in the moderate shock state and so on. Thus if the household faces deprivation over all the different states, the average income in the extreme shock state would be lower than the average income in the moderate shock state which in turn would be lower than the mild state. It is possible, however, that the household just faces moderate shocks among the negative shocks. In such case, there would be no income to report for extreme shocks or for mild shocks. Due to informational constraints, we have framed the different states in this nested manner, however, our theoretical model provides a general framework where there is no need for such restrictions. For instance, if we can classify the income based on different shocks such as health shocks, or unemployment shocks or natural shocks (flood or drought), then there is no reason why these different states should be nested in the way that we see in our application.

We use a simple frequency based rule to calculate probabilities associated with the different states. Zhang and Wan (2008) find that relative frequency is equally good in predicting vulnerability to poverty. For any household the probability of a state is the total number of weeks the income was in that range, divided by the total number of weeks the household earned income. Thus, each household has a different probability distribution over the states depending on the weeks it has earned the incomes in the different ranges. Note that individuals in the same household will have the same probabilities of the states. In assessing the frequencies of different states we have taken all available information into account up till that point. For 2017, we have taken the frequency information of 2017, 2016 and 2015 if those were available for the individuals. Similarly, for the frequency in 2018, we have considered information from 2018 to 2015. Thus for 2017 and 2018 on an average for each household, we have considered 80 and 132 weeks of information respectively to calibrate the probabilities of the different states of nature. Unlike deprivation, the probabilities of the states are not nested. Hence, for instance, moderate income shocks may have a lower probability than either the extreme and the mild shock states.

To measure societal vulnerability we proceed in three steps. First, for the identification step, we follow the *P-rule*, given by $\rho^E(L)$, referred previously. We can now decide the level of first cut-off (θ) of this full vulnerability. Individuals below θ are not vulnerable enough to be considered in our analysis, and hence assigned a vulnerability level of zero. Next, we compute the weighted deprivation for each individual who is identified as vulnerable. This is the vulnerability to poverty that each individual faces. Third, since each member of a household has the same level of vulnerability, we multiply the individual vulnerability with the household size to get the total vulnerability of that household. The average of the vulnerability of all the individuals in the society is the societal vulnerability.

In our empirics we choose several different cut-off levels to assess the vulnerability. We start with $\theta = \theta_{\min}$, the 'union' approach, where an individual is deemed vulnerable if they have a positive probability to be in deprived in the future. Further we consider $\theta = 0.25$, $\theta = 0.5$, $\theta = 0.75$ and $\theta = 1$. For measuring the level of vulnerability we take $\alpha = 1$. However, it can be easily extended to other values of α. What is important to keep in mind is that with the cutoffs, the pool of vulnerable individuals keep changing and not the states. Thus an extremely deprived individual will have the same vulnerability whether $\theta = \theta_{\min}$, or $\theta = 0.5$. The computation of vulnerability for different levels of θ for 2017 and 2018 is presented in Table 1 below:

Table 1: Levels of Vulnerability in Hrishipada, Bangladesh: 2017-18

(1)2-5 (1)6-9	2017				2018			
			Head Count				Head Count	
	Societal Vulnerability	Intensity of Vulnerability	Ratio of Vulnerable Households	Median Vulnerability	Societal Vulnerability	Intensity of Vulnerability	Ratio of Vulnerable Households	Median Vulnerability
$\theta = \theta_{min}$	0.501	0.519	0.966	0.537	0.496	0.523	0.948	0.512
$\theta = 0.25$	0.480	0.584	0.822	0.537	0.475	0.582	0.817	0.512
$\theta = 0.50$	0.421	0.660	0.638	0.537	0.416	0.644	0.645	0.512
$\theta = 0.75$	0.225	0.770	0.292	0.000	0.207	0.777	0.267	0.000
$\theta = 1$	0.017	0.830	0.020	0.000	0.004	0.900	0.004	0.000
Standard Approach	0.386	0.690	0.560	0.537	0.360	0.684	0.526	0.512

There are several interesting results that emerge from the calibrations. First, although based on a small specific sample, our estimates are not too off the mark for Bangladesh. The World Bank (2017) country profile for Bangladesh shows that 85 percent of the country would be considered poor if we take the cut-off to be 3200 taka per week (or \$5.50 per day) as we have done here. Our results show that for the 'union' approach ($\theta = \theta_{\min}$), on an average 95 percent of the sample is vulnerable to poverty.[14] Given that this is a rural sample focussed on people participating in a micro-saving scheme, the high proportion of vulnerable individuals that we see in our sample is not surprising.

Second, for both 2017 and 2018, there is a clear trend of societal vulnerability going down as θ increases. This is because progressively we are considering a smaller set of individuals in our vulnerability calculations. When we consider the 'union' approach ($\theta = \theta_{\min}$) for 2017, the vulnerability

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[14]Azam and Imai (2009) find that for rural areas in 2005 the proportion of the vulnerable is 52 percent. In comparison, under our standard approach (which is broadly the method used by Azam and Imai (2009)), we find on an average around 54 percent of our sample is vulnerable with an overall societal vulnerability to be around 0.50.

level is 0.50, which is considerably high, with 96 percent of the individuals classified as vulnerable. However, as θ goes up, there is not much reduction in the level of vulnerability index initially. If we focus on $\theta = 0.5$, which is the cut-off considered by many empirical studies, societal vulnerability is 0.42, which is a significant reduction of 16 percent from $\theta = \theta_{\min}$. As we move to $\theta = 0.75$ the overall index reduces by more than 50 percent.

Third, based on equation (7) we decompose societal vulnerability for both 2017 and 2018, into the head-count ratio and the intensity of vulnerability. What is evident is that these two aspects work in the opposite direction with the change in θ. Thus as θ increases, the head-count ratio decreases, whereas the intensity of vulnerability increases. Given that the vulnerability index does not change much with θ, till we reach $\theta = 0.5$ it is not surprising that both head-count and intensity also shows sharp changes around those values of θ. What is interesting is that the reduction in the head-count dominates the increase in the intensity. For instance in 2017, as we move from $\theta = 0.5$ to the next cut-off of $\theta = 0.75$, the intensity increases by 17 percent, the head-count drops by around 50 percent. It is this large decrease in the headcount which leads to a reduction in vulnerability despite the increase in the intensity of vulnerability.

Fourth, if we compare the standard approach in the literature to our method, we get very different results. Under the standard approach, vulnerability for each household is calibrated based on the 'union' criteria (i.e. $\theta = \theta_{\min}$) and then a cut-off of 0.5 is applied. Thus, households with vulnerability less than 0.5, are not considered vulnerable. If we apply that method to our data, (Row 6 of Table 1), we find that for 2018 the overall societal vulnerability is 0.36 and around 53 percent of the population is vulnerable. This is in sharp contrast to our method based on a $\theta = 0.5$ cut-off, where the head-count is 65 percent and societal vulnerability is 0.42.[15] In this context, the standard approach underestimates both the incidence and the level of vulnerability between 15 to 20 percent. We see similar underestimation of the incidence and level of vulnerability under the standard approach for 2017 too.

Another advantage of our framework is that we can decompose the overall vulnerability to investigate which shocks are contributing most to the societal vulnerability. Equation (8) shows how societal vulnerability V^S can be decomposed as the sum of the vulnerabilities arising from different shocks. The relative contribution of a shock m to societal vulnerability is V^m/V^S, where V^m is based on (9). We present the relative contribution (in percentage) of extreme shocks, moderate shocks and mild shocks to societal vulnerability for 2017 and 2018 in the table below for different values of θ.

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[15] Note that the cut-off in the standard approach differs from ours in the sense that we only consider the probability of deprivation in the future for identification, whereas in the standard case, both the probability and depth is taken in to account.

Table 2: Decomposition of Vulnerability based on Shocks: 2017-2018

(1)2-4 (1)5-7	2017			2018		
	Extreme Shocks	Moderate Shocks	Mild Shocks	Extreme Shocks	Moderate Shocks	Mild Shocks
$\theta = \theta_{min}$	51.13	20.28	28.59	47.97	21.70	30.33
$\theta = 0.25$	51.26	20.56	28.18	46.94	22.89	30.17
$\theta = 0.50$	50.68	21.78	27.54	46.60	24.78	28.62
$\theta = 0.75$	53.89	24.10	22.01	52.80	28.57	18.63
$\theta = 1$	86.41	0.00	13.59	100.00	0.00	0.00

For all the different cut-off levels, for both years, around 50 percent of the societal vulnerability is arising from extreme shocks. To some extent, one can argue that this is because of the nested nature of the deprivation, where extreme shocks imply a higher deprivation compared to other shocks. Interestingly, however, for both years, the next biggest contributor to societal vulnerability is mild shocks for the majority of the cases. Given the nested nature of the deprivation, moderate shocks are associated with a higher level of future deprivation compared to mild shocks, and thus, the only way vulnerability under mild shocks can be higher compared to moderate shocks is if the incidence of mild shocks is significantly greater relative to moderate shocks. What this tells us is a lack of consumption smoothing since individuals have a high probability of either falling a little short of the poverty line or falling massively below the poverty line. Note also that the contribution of the different shocks is non-monotonic in the different cut-offs. For instance with 2017, as we move from $\theta = \theta_{min}$ to $\theta = 0.25$, the contribution of extreme shocks increases, however, as we move further to $\theta = 0.5$ the contribution of extreme shocks decline. More broadly, if we had more specific knowledge of the type of shocks, such as health, natural, employment shocks, through this decomposition, we would have been able to point to the source of vulnerability more clearly. Any policy to reduce vulnerability could then build on such information.

7   Conclusion

The main innovation of the paper is in bringing a clear identification part to the measurement of vulnerability as has been done in the literature on poverty and multi-dimensional deprivation. Identification in measuring vulnerability is a highly active area of research and our paper contributes to that growing literature. As in the case of income poverty, we have argued that there can be multiple

identification methods for vulnerability, each of which may have their own normatively appealing properties. We have provided characterisation of three possible identification rules - one in which all the states with positive deprivation are weighted according to their probability and two other rules based on different levels of informational requirements (see Appendix B). Exploring vulnerability from this perspective also allows us to link the issue to the well-established methods of multidimensional deprivation. The analytical structure of the problems is very similar. In the multidimensional context individuals face deprivations across different dimensions. In this paper, we have argued that a natural way to conceptualise the idea of vulnerability to poverty is to think in terms of the adverse income shocks that individuals face, which can be captured in terms of the different states of nature. Hence, identification in our context is essentially the (weighted) number of income shocks in the future the person might face.

This leads us to the next issue of measuring the level of vulnerability of those individuals identified as vulnerable. Here we axiomatically characterise a popular class of individual vulnerability measures which are based on the expected FGT class of poverty measures. This class of measures has also been used as an identification rule, since in many empirical studies, identification and aggregation are not distinct exercises. Interestingly, if we allow the identification rule to significantly differ from the aggregation step, i.e. measurement of the level of vulnerability, it may lead to inconsistencies in choice of the set of vulnerable individuals. Such inconsistencies are predicated on the assumption that we have the same information for identifying vulnerable individuals as we have to measure their level of vulnerability. As we have argued strongly in this paper, that position is not realistic from a practical perspective.

Although we provide a real-world application of our proposed methods to identify and measure vulnerability, it is mainly to demonstrate how the *P-rule* identification strategy can be applied systematically. In our data, from a hamlet in Bangladesh, we find that our proposed method, based on strong axiomatic justifications, can differ from the standard method used in the literature by 15 to 20 percent. This difference arises because to identify the vulnerable our approach considers a limited set of information that most policymakers and practitioners would face in the real world, compared to a much richer set of information that is required under the standard approach. One advantage of our approach is that we are able to demonstrate how much of the societal vulnerability is contributed by each of the various shocks. In our empirical illustration, we found the majority of the vulnerability was arising from 'extreme' shocks and may be related to chronic nature of poverty faced by most the households in our sample. In a more general setting, where vulnerability arises from various shocks such as health, economic or natural shocks, this ability to decompose vulnerability in terms of the contribution of the shocks, can be extremely useful from a policy perspective.

If we are serious about reducing vulnerability, then identification has to be an important and distinct part of our measurement of vulnerability. A realistic approach towards identifying the vulnerable also has to keep in mind the substantial informational challenges in identifying the vulnerable and measuring the level of vulnerability. This paper is a step towards addressing both these challenges, with the hope that it will lead to better-targeted policies to reduce vulnerability.

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A Appendix

Proof of Proposition 1: Only if. Suppose ρ satisfies axioms 2 and 3. Then given any two lotteries L and L' we show the following:

Case I:

$$\mathbf{p}^L \cdot \mathbf{r} > \mathbf{p}^{L'} \cdot \mathbf{r} \implies \rho(L) > \rho(L') \ , \tag{E-A1}$$

We apply the method of induction. Let n stand for the number of deprived states with positive probability.

Suppose $n = 1$. Consider two lotteries L_1 and L'_1. Suppose for L_1 the deprived state with positive probability is s and for L'_1 it is s'. In this case

$$\mathbf{p}^L \cdot \mathbf{r} > \mathbf{p}^{L'} \cdot \mathbf{r} \implies p_s^L > p_{s'}^{L'}$$

where p_s^L is the probability of state s in lottery L. Then from axiom 2 we can show that

$$p_s^L > p_{s'}^{L'} \implies \rho(L) > \rho(L') \ .$$

Hence for $n = 1$, (E-A1) is satisfied.

Now suppose for $n = k < m$ states too (E-A1) is satisfied. This implies that for any lottery L and L'

$$\sum_{n=1}^{k} p_n^L r^n > \sum_{n=1}^{k} p_n^{L'} r^n \implies \rho(L) > \rho(L'). \tag{E-A2}$$

Now consider $n = k + 1 \leq m$, we have to show that

$$\sum_{n=1}^{k+1} p_n^L r^n > \sum_{n=1}^{k+1} p_n^{L'} r^n \implies \rho(L) > \rho(L').$$

Consider any lottery, $L(k+1) = (p_1, d^1; p_2, d^2; ...; p_k, d^k; p_{k+1}, d^{k+1}; ...; p_m^L, d^m)$ such that (i)$\forall s$, $s < k + 1$, $p_s > 0$ and $d^s > 0$ and (ii) $\forall s$, $s > k + 1$, $p_s = 0$ or $d^s = 0$. Now construct a lottery $\widehat{L}(k)$ from $L(k+1)$ such that $\widehat{L}(k) : (p_1^L, d^1; p_2^L, d^2; ...; \widehat{p}_k, d^k; 0, d^{k+1}; ...; p_m^L, d^m)$, where $\widehat{p}_k = (p_k^L + p_{k+1}^L)$. Then from axiom 3, $\rho(L(k+1)) = \rho(\widehat{L}(k))$. Suppose by repeated transfer of probability we arrive at $\widetilde{L}(1) : (\widetilde{p}_1, d^1; 0, d^2; ...; 0, d^m)$, where $\widetilde{p}_1 = \sum_{n=1}^{k+1} p_n$. From axiom 3 we can derive,

$$\rho(L(k+1)) = \rho(\widetilde{L}(1)). \tag{E-A3}$$

Similarly for a lottery $L'(k+1) : (p'_1, d^1; p'_2, d^2; ...; p'_k, d^k; p'_{k+1}, d^{k+1}; ...; p'_m, d^m)$ through repeated transfer of probability we can arrive at $\widetilde{L}'(1) : (\widetilde{p}'_1, d^1; 0, d^2; ...; 0, d^m)$, where $\widetilde{p}'_1 = \sum_{n=1}^{k+1} p'_n$. From axiom

3 we know

$$\rho(L'(k+1)) = \rho(\widetilde{L}(1)). \tag{E-A4}$$

Suppose, without loss of generality, $p_1^{\widetilde{L}} \geq p_1^{\widetilde{L}'}$, then using the axiom 2, (E-A3) and (E-A4)we can claim

$$
\begin{aligned}
p_1^{\widetilde{L}} &> p_1^{\widetilde{L}'} \implies \rho(\widetilde{L}(1)) > \rho(\widetilde{L}'(1)), \\
\sum_{n=1}^{k+1} p_n &> \sum_{n=1}^{k+1} p_n' \implies \rho(L(k+1)) > \rho(L'(k+1)).
\end{aligned}
$$

Case II:

$$\mathbf{p}^L \cdot \mathbf{r} = \mathbf{p}^{L'} \cdot \mathbf{r} \implies \rho(L) = \rho(L') \ .$$

The proof is similar to Case I and is omitted.

It can be easily checked that the sufficient conditions are satisfied. ∎

Before we present the proof of Proposition 2, we show that the individual vulnerability function as given in equation (1), is continuous. We would need this result to prove Proposition 2.

Lemma A.1 *Axiom of Decomposability (A5) implies that the vulnerability function $V(L)$ is continuous in $\mathbf{p}^L$.*

Proof of Lemma A.1: We demonstrate the result by contradiction using a diagrammatic proof. Given Axiom of Decomposability (A5), $V(L)$ is an affine function. Suppose, $V(L)$ is discontinuous at L^3 where $L^3 = (\pi_1, d^1; (1-\pi_1), 0)$. Now consider two lotteries, $L^1 = (p_1, d^1; (1-p_1), 0)$ and $L^2 = (p_1', d^1; (1-p_1'), 0)$ where without loss of generality, $V(L^2) > V(L^1)$. Note that lotteries with two states can be represented in a line. From Figure 1, below we can see that $\alpha V(L^1) + (1-\alpha)V(L^2)$ will lie on the line AB, for different values of α.

Let for $0 < \alpha = \alpha^* < 1$, $L^3 = \alpha^* L^1 + (1-\alpha^*)L^2$, i.e. $\pi_1 = \alpha^* p_1 + (1-\alpha^*)p_1'$. Then we can easily establish from Figure 1, given the discontinuity at L^3, that $\alpha^* V(L^1) + (1-\alpha^*)V(L^2) < V(\alpha^* L^1 + (1-\alpha^*)L^2)$, thus violating Axiom of Decomposability (A5). ∎

Proof of Proposition 2: We start by proving the necessary condition. For any lottery $L = (p_1, d^1; ..; p_m, d^m)$ faced by an individual where $\mathbf{p}^L \cdot \mathbf{r} \geq \theta$. From Lemma 1 of Dutta et al. (2011) we know that Axiom A5 implies that

$$V(L) = \sum_{s=1}^{m} p_s f(d^s). \tag{E-A5}$$

Consider two lotteries $L = (p_1, d^1; ...; p_{k'}, d^{k'}; p_k, d^k; ...; p_m, d^m)$ and $L' = (p_1, d^1; ...; p_{k'} + \delta, d^{k'}; p_k - \delta, d^k; ...; p_m, d^m)$, such that $p_{k'} > 0$; $p_k > 0$, $d^{k'} \geq d^k > 0$. Then using A5 and (E-A5) and cancelling terms, we can show

$$p_{k'} f(d^{k'}) + p_k f(d^k) < (p_{k'} + \delta) f(d^{k'}) + (p_k - \delta) f(d^{k'})$$

$$\implies f(d^k) \leq f(d^{k'}). \tag{E-A6}$$

Given that $d^{k'} \geq d^k$, and (E-A6) holds for any arbitrary k' and k, one can infer that $f(d^s)$ is monotonic. From Lemma A.1 we know that $V(L)$ is continuous. Given (E-A5) and $p \in [0,1]$, Lemma A.1 implies that that $f(d^s)$ must be continuous.

Given Axiom A4 for any lottery L' such that $\mathbf{p}^{L'} \cdot \mathbf{r}^L < \theta$, thus, $V(L') = 0$.

Now consider lotteries, $L = (p_1, d^1; ...; p_m, d^m)$, $L' = (p_1, d^1; ...; p_k + \delta, d^k; p_l - \delta, d^l; ...; p_m, d^m)$, $\widetilde{L} = (\widetilde{p}_1, d^1; ...; \widetilde{p}_s, d^s; \widetilde{p}_t, d^t; ...; \widetilde{p}_m, d^m)$, $\widetilde{L}' = (\widetilde{p}_1, d^1; ...; \widetilde{p}_s + \delta, d^s; \widetilde{p}_t - \delta, d^t; ...; \widetilde{p}_m, d^m)$, where $d^k \geq d^l$ and $d^s \geq d^t$, $p_l - \delta > 0$ and $\widetilde{p}_t - \delta > 0$, such that $V(\widetilde{L}') - V(\widetilde{L}) = V(L') - V(L)$. Then Axiom A7 implies $V(\widetilde{L}'_\lambda) - V(\widetilde{L}_\lambda) = V(L'_\lambda) - V(L_\lambda)$, $0 < \lambda < 1$, where $L_\lambda = (p_1, \lambda d^1; ...; p_m, \lambda d^m)$. Thus, using (E-A5), we can show that for all $d^k, d^l, d^s, d^t \in (0,1]$

$$[f(d^k) - f(d^l) = f(d^s) - f(d^t)] \implies [f(\lambda d^k) - f(\lambda d^l) = f(\lambda d^s) - f(\lambda d^t)]. \tag{E-A7}$$

Following similar steps as in Chakraborty et al. (2008) , for any $d^j, d^k \in (0,1]$ such that $d^j \geq d^k$ (E-A7) can be written as,

$$f(\lambda d^j) - f(\lambda d^k) = h(f(d^j) - f(d^k), \lambda). \tag{E-A8}$$

Suppose for $d^j \geq d^k \geq d^l$, $u = f(d^j) - f(d^k)$ and $v = f(d^k) - f(d^l)$. Then

$$h(u + v, \lambda) = h(u, \lambda) + h(v, \lambda),$$

which is a Cauchy equation and whose solution is $h(u, \lambda) = g(\lambda)u$. Thus from (E-A8) we can get

$$f(\lambda d^j) - f(\lambda d^k) = g(\lambda)(f(d^j) - f(d^k)),$$

which implies

$$f(\lambda d^j) - g(\lambda)f(d^j) = f(\lambda d^k) - g(\lambda)f(d^k). \tag{E-A9}$$

Since $f(d)$ is monotonic and continuous and (E-A9) should hold for all d^j and d^k, therefore it must

be the case that for all d,

$$f(\lambda d) = g(\lambda) f(d) \tag{E-A10}$$

Suppose $d = 1$, given $f(1) = 1$, it implies that $f(\lambda) = g(\lambda)$. Replacing this in (E-A10) we get

$$f(\lambda d) = f(\lambda) f(d). \tag{E-A11}$$

(E-A11) is a Cauchy equation, whose general solution, for $d > 0$, is given by (Aczel 1966)

$$f(d) = d^\alpha \tag{E-A12}$$

where $\alpha \geq 0$. Thus from (E-A5) and (E-A12) we can show $V(L) = \sum_{s=1}^{m} p_s (d^s)^\alpha$, $\alpha \geq 0$. Further it can be easily checked that the sufficient conditions are satisfied. ∎

B Appendix

In this Appendix we characterise two identification rules: (a) counting rule and (b) max rule.

Counting-rule: We provide a characterisation of an identification rule based on the counting method where just the number of states in the future the person is deprived is known. Consider a lottery $L = (p_1^L, d^1; p_2^L, d^2; ...; p_m^L, d^m)$. We define an associated vulnerability identification vector $\widehat{\mathbf{r}}^L = \{\widehat{r}_L^1, \widehat{r}_L^2, ..., \widehat{r}_L^m\}$ where,

$$\forall s, \widehat{r}_L^s = \begin{cases} 1 & \text{if } p_s^L . d^s > 0 \\ 0 & \text{otherwise} \end{cases}.$$

This distinguishes the states which are deprived with positive probabilities. We define counting function as, ρ^C as

Definition B-1 ρ^C *is a counting function such for any two lotteries L and L': (i) $\rho^C(L) > \rho^C(L')$ iff $\left\|\widehat{\mathbf{r}}^L\right\|^2 > \left\|\widehat{\mathbf{r}}^{L'}\right\|^2$ and (ii) $\rho^C(L) = \rho^C(L')$ iff $\left\|\widehat{\mathbf{r}}^L\right\|^2 = \left\|\widehat{\mathbf{r}}^{L'}\right\|^2$.*

This is a pure counting function, where we differentiate the two lotteries based on the number of deprived states with positive probabilities. We characterise the counting function using the following axioms.

Axiom B1 *Monotonicity of deprived states. Let $L = (p_1^L, d^1; p_2^L, d^2; ...; p_{k+1}^L, d^{k+1}, ...p_m^L, d^m)$, where $\forall s = 1, ..., k+1$, $\widehat{r}_L^s > 0$ and $\forall s > k+1$, $\widehat{r}_L^s = 0$, and $L' = (p_1^{L'}, d^1; p_2^{L'}, d^2; ...; p_{k+1}^{L'}, d^{k+1}, ...p_m^{L'}, d^m)$, where $\forall s = 1, ..., k$, $\widehat{r}_{L'}^s > 0$ and $\forall s > k$, $\widehat{r}_{L'}^s = 0$. Then $\rho(L) > \rho(L')$.*

Axiom B2 *Strong Invariance to probability transfers.* Let $L = (p_1^L, d^1; p_2^L, d^2; ...; p_k^L, d^k, ...p_m^L, d^m)$, where $\hat{r}_L^k > 0$, and $L' = (p_1^L, d^1; p_2^L, d^2; ...; p_k^L - \epsilon, d^k, ..., p_{k'}^L + \epsilon, d^{k'}, ..., p_m^L, d^m)$, where $p_k^L > \epsilon$. Then $\rho(L) = \rho(L')$.

The intuition of the first axiom is that if a lottery has more deprived states with positive probability than another, it should have a higher value than the other lottery in terms of the identification rule.

The second axiom captures the idea that if you transfer some probability from a deprived state to any other state, then the counting function remains unchanged.

Proposition B.1 *A counting function ρ satisfies Axioms of Monotonicity of Deprived States (B1) and Axiom of Strong Invariance to Probability Transfers (B2) iff $\rho = \rho^C$.*

Sketch of the Proof: Only if: Consider any two lotteries L and L'. We will consider two cases: (I) where the number of deprived states with positive probability is different for the two lotteries and (II) where it is same.

Case I: Without loss of generalisation suppose, $\left\|\hat{r}^L\right\|^2 > \left\|\hat{r}^{L'}\right\|^2$, then clearly from Axiom B1, we can conclude that $\rho(L) > \rho(L')$.

Case II: Now suppose $\left\|\hat{r}^L\right\|^2 = \left\|\hat{r}^{L'}\right\|^2$.

(i) Let the number of deprived states with positive probability be $n < m$. Suppose $\forall s = 1, ..., n$, $p_s^L \geq p_s^{L'}$, with strict inequality holding for some states, then it must be the case that $\forall s = n, ..., m$, $p_s^L \leq p_s^{L'}$ with strict inequality for some states. Then applying Axiom B2, repeatedly on L, where we transfer probability from deprived to non-deprived states we can retrieve L'. Thus $\rho(L) = \rho(L')$. On the other hand if $\exists s \in \{1, ..., n\}$, $p_s^L > p_s^{L'}$ and $\exists s' \neq s \in \{1, ..., n\}$, $p_{s'}^L < p_{s'}^{L'}$. Then again through repeated application of Axiom B2 on L, where we move probabilities to deprived and non deprived states, we can retrieve L' and thus $\rho(L) = \rho(L')$.

(ii) Let $n = m$, then $\exists s \in \{1, ..., m\}$, $p_s^L > p_s^{L'}$ and $\exists s' \neq s \in \{1, ..., m\}$, $p_{s'}^L < p_{s'}^{L'}$. By repeated application of Axiom B2, where we move probabilities between deprived states, we can derive L' from L which implies $\rho(L) = \rho(L')$.

The If part of the proof is straight forward and hence omitted.∎

This characterises the counting function which is purely based on the number of deprived states with positive probability.

Max-rule: In this part we provide a characterisation of an identification rule based on maximum deprivation the person will face in the future. Let $\mathcal{L}$ be the set of all possible lotteries faced by an individual where a lottery $L = (p_1^L, d^1; p_2^L, d^2; ...; p_m^L, d^m)$. The individual has a complete and transitive preference ordering $\succsim$ over $\mathcal{L}$. We define an associated vulnerability identification vector

$\widehat{\mathbf{r}}^L = \{\widehat{r}_L^1, \widehat{r}_L^2, ..., \widehat{r}_L^m\}$ where:

$$\forall s, \ \widehat{r}^s = \begin{cases} d^{\max} & \text{if } p_s.d^s > 0 \\ 0 & \text{otherwise} \end{cases} .$$

where $d^{\max} = \max(d^1, d^2, ..., d^j)$ such that for all $s = 1, ..., j$, $p_s.d^s > 0$. It is clear that $\mathbf{r}^L$ basically partitions the states into deprived and the non-deprived, such that all the deprived states have a positive probability of occurrence and have been assigned the highest deprivation.

A max identification function $\rho^{\max}$, is defined as follows:

Definition B-2 $\rho^{\max} : \mathcal{L} \longrightarrow \mathbb{R}_+$, *is a vulnerability identification function such that for any individual (i)* $\rho^{\max}(L) > \rho^{\max}(L')$ *iff* $\mathbf{p}^L \bullet \widehat{\mathbf{r}}^L > \mathbf{p}^{L'} \bullet \widehat{\mathbf{r}}^{L'}$ *(ii)* $\rho^{\max}(L) = \rho^{\max}(L')$ *iff* $\mathbf{p}^L \bullet \widehat{\mathbf{r}}^L = \mathbf{p}^{L'} \bullet \widehat{\mathbf{r}}^{L'}$.

To characterise the max identification rule we propose the following axioms:

Axiom B3 *Axiom of Independence of Irrelevant states: Consider lotteries* $L = (p_1, d^1; ...; p_k, d^k; ...; p_m, d^m)$, *where* $p_k.d^k > 0$ *and* $\forall s \neq k$, $p_s.d^s = 0$, $L' = (p'_1, d^1; ...; p_{k'}, d^{k'}; ...; p'_m, d^m)$, $\forall s \neq k'$, $p'_s.d^s = 0$, *if* $p_k = p_{k'}$, *then* $\rho(L) = \rho(L')$ *iff* $d^k = d^{k'}$.

Axiom B4 *Axiom of Single State Deprivation: Consider lotteries* $L = (p_1, d^1; ...; p_k, d^k; ...; p_m, d^m)$, *where* $p_k \geq 0$, $d^k > 0$ *and* $\forall s \neq k$, $p_s.d^s = 0$, $L' = (p'_1, d^1; ...; \delta, d^k; ...; p'_m, d^m)$, $\delta \geq 0$ *and* $\forall s \neq k$, $p'_s.d^s = 0$ *and* $L'' = (p''_1, d^1; ...; p_k + \delta, d^k; ...; p''_m, d^m)$, $\forall s \neq k$, $p''_s.d^s = 0$, *then* $\rho(L'') = \rho(L) + \rho(L')$.

Axiom B5 *Axiom of Weak Monotonicity: Consider two lotteries* $L = (p_1, d^1; ...; p_k, d^k; ...; p_{k'}, d^{k'}; ...; p_m, d^m)$ *and* $L' = (p_1, d^1; ...; p_k + \delta, d^k; ...; p_l - \delta, d^l; ...; p_m, d^m)$, $\delta > 0$ *such that* $p_k > 0$; $p_l - \delta \geq 0$, $d^k \geq d^l > 0$. *Then* $\rho(L') \geq \rho(L)$.

Our first axiom broadly captures the intuition that states where deprivation does not take place with positive probability should not matter. Thus if there is a state where one would be deprived but it never happens, then it should not matter in our measurement of vulnerability. The second axiom is crucial. Suppose there is only one future state with positive probability of deprivation, then according to the axiom, for identification purposes we need to take both the probability and the deprivation level into account. The third axioms implies that if for a lottery the probability of high deprivation state increases while that of low deprivation state decreases, then vulnerability should increase. Given these three axioms we can show the following result.

Proposition B.2 *An identification function ρ satisfies Axioms of Independence of Irrelevant States (B3), Single State Deprivation (B4) and Axiom of Weak Monotonicity (B5) iff $\rho = \rho^{\max}$.*

Proof: Only if. We proceed by induction. Consider a lottery $L = (p_1, d^1; ..., p_k, d^k; ...; p_l, d^l; ...; p_m, d^m)$, where $d^1 \geq d^2 \geq ... \geq d^m$. Let t represents the total number of states in lottery L such that $p_s.d^s > 0$.

Consider lotteries, $L = (p_1, d^1; ...; p_l, d^l; ...; p_m, d^m)$ where $\forall s \neq l$, $p_s.d^s = 0$, $d^l > 0$ and $p_l \geq 0$; $L' = (p'_1, d^1; ...; \delta, d^l; ...; p'_m, d^m)$, $\delta \geq 0$; and $L'' = (p''_1, d^1; ..., p_k + \delta, d^k; ...; p''_m, d^m)$. Then, from Axiom B3, we can write $\rho(L) = F(p_l, d^l)$, $\rho(L') = H(\delta, d^l)$ and $\rho(L'') = G(p_l + \delta, d^l)$. Applying Axiom B4, we can arrive at

$$G(p_l + \delta, d^l) = F(p_l, d^l) + H(\delta, d^l). \tag{E-B1}$$

This is a Pexider equation. Given that by definition, $\rho(L') = 0$, when $\delta = 0$ and $\rho(L) = 0$ when $p_l = 0$, from Aczel (1966) we can write (E-B1) as

$$G(p_l, d^l) = p_l.g(d^l). \tag{E-B2}$$

Since l is arbitrary, this holds true for all states. Further applying Axiom B5, we can show that for any two states k, l, $g(d^k) \geq g(d^l)$, whenever $d^k \geq d^l$. Now consider lottery $L = (p_1, d^1; ...; p_k, d^k; ...; p_m, d^m)$, where $p_k.d^k > 0$ and $\forall s \neq k$, $p_s.d^s = 0$, and lottery $L' = (p'_1, d^1; ...; p_k, d^{k'}; ...; p'_m, d^m)$, $d^k > 0$, $\forall s \neq k'$, $p'_s.d^s = 0$. Applying Axiom B3, we know that if $\rho(L) = \rho(L')$, which from (E-B2) implies, $g(d^k) = g(d^{k'})$, then it must be the case that $d^k = d^{k'}$. Similarly we can show that if $d^k = d^{k'}$, then $g(d^k) = g(d^{k'})$. Thus it must be the case that,

$$g(d^k) > g(d^{k'}) \text{ whenever } d^k > d^l \text{ and } g(d^k) = g(d^{k'}), \text{whenever } d^k = d^l. \tag{E-B3}$$

Note for any lottery L, with $t = 1$, suppose $p_s.d^s > 0$ then $d^s = \widehat{r}^s$. Hence, $\rho(L') > \rho(L) \implies \rho^{\max}(L') > \rho^{\max}(L)$, and $\rho(L') = \rho(L) \implies \rho^{\max}(L') = \rho^{\max}(L)$.

Now suppose for

$$t < k, \ \rho(L') > \rho(L) \implies \rho^{\max}(L') > \rho^{\max}(L) \text{ and } \rho(L') = \rho(L) \implies \rho^{\max}(L') = \rho^{\max}(L). \tag{E-B4}$$

We show that (E-B4) when $t = k$. Let $L = (p_1, d^1; ..., p_k, d^k; ...; p_m, d^m)$ such that $\forall s > k$, $p_s.d^s = 0$. Consider a lottery where $L' = (p_1 + p_{k'}, d^1; ..., 0, d^k; ...; p_m, d^m)$. From Axiom B5 we know

$$\rho(L') \geq \rho(L)$$

Note that L' has $(k-1)$ states where $p_s.d^s > 0$. Hence using (E-B4) we can derive

$$\sum_{s=1}^{k-1} p_s.g(d^1) = \rho(L') \geq \rho(L) \tag{E-B5}$$

Now consider another lottery $L'' = (p_1, d^1; ...; 0, d^{k'}; ...; p_k + p_{k'}, d^k; ...; p_m, d^m)$. Applying Axiom B5, we can write

$$\rho(L) \geq \rho(L'').$$

Note that L'' has $(k-1)$ states where $p_s.d^s > 0$. Hence as before applying (E-B4) we can derive

$$\rho(L) \geq \rho(L'') = \sum_{\substack{s=1 \\ s \neq k'}}^{k} p_s.g(d^1) \tag{E-B6}$$

By construction it is the case that $\sum_{\substack{s=1 \\ s \neq k'}}^{k} p_s = \sum_{s=1}^{k-1} p_s$, then comparing (E-B5) and (E-B6) we can arrive at

$$\rho(L) = \sum_{s=1}^{k} p_s.g(d^1) = \sum_{s=1}^{k} p_s.g(r^s). \tag{E-B7}$$

Given (E-B3), from (E-B7) one can establish that $\rho(L') > \rho(L) \implies \rho^{\max}(L') > \rho^{\max}(L)$ and $\rho(L') = \rho(L) \implies \rho^{\max}(L') = \rho^{\max}(L)$. Further, it can be shown that $\rho^{\max}(L)$ satisfies axioms B3, B4 and B5.∎

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